A TOWER CONNECTING GAUGE GROUPS TO STRING TOPOLOGY

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Abstract. We extend a recent result of Cohen and Jones [12] relating the gauge group $\mathcal{G}(\mathcal{P})$ of a principal bundle $\mathcal{P}$ over M to the Thom ring spectrum $(\mathcal{P}^{\mathrm{Ad}})^{-TM}$. If $\mathcal{P}$ has contractible total space, the resulting Thom ring spectrum is LM^{-TM}, which plays a central role in string topology. Cohen and Jones show that $(\mathcal{P}^{\mathrm{Ad}})^{-TM}$ is the linear approximation of $\mathcal{G}(\mathcal{P})$ in a certain sense, and we extend that relationship by demonstrating the existence of higher-order approximations and calculating them explicitly. This also generalizes calculations done by Arone in [1].

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1. Introduction

If M is a closed oriented manifold and $LM = \mathrm{Map}(S^1, M)$ is its free loop space, then the homology $H_*(LM)$ has a *loop product* first described by Chas and Sullivan [5]. This loop product is homotopy invariant [15] and has been calculated in a number of examples [13]. In [18], Félix and Thomas studied the loop product by defining a multiplication-preserving map

$$H_*(\Omega_{\mathrm{id}}\mathrm{haut}(M);\mathbb{Q}) \longrightarrow H_{*+\dim M}(LM;\mathbb{Q}) \tag{1}$$

where $\mathrm{haut}(M)$ is the space of self-homotopy equivalences of M, and the loops are based at the identity map of M.

In [11], Cohen and Jones described a ring spectrum LM^{-TM} whose homology is $H_*(LM)$ but with a grading shift. The multiplication on LM^{-TM} gives the loop product on $H_*(LM)$, and that the map of Félix and Thomas (1) comes from a map of ring spectra

$$\Sigma^\infty_+\Omega_{\mathrm{id}}\mathrm{haut}M \longrightarrow LM^{-TM} \tag{2}$$

by taking rational homology groups [12]. In the forthcoming paper [12], Cohen and Jones extend this map of ring spectra to a *natural transformation of functors*

$$F \longrightarrow L \tag{3}$$

$$F, L : \mathcal{R}_M^{\mathrm{op}} \longrightarrow \mathcal{S}p$$
$$F(M \amalg M) = \Sigma^\infty_+\Omega_{\mathrm{id}}\mathrm{haut}M$$
$$L(M \amalg M) \simeq LM^{-TM}$$

Here $\mathcal{R}_M$ is the category of retractive spaces over M and $\mathcal{S}p$ is the category of spectra. We will give these functors explicitly in section 3. Both F and L are required to be *homotopy functors*, meaning that they send equivalences of spaces to equivalences of spectra. Cohen and Jones show that L is the universal approximation of F by an *excisive* homotopy functor, i.e. one that takes each homotopy pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

to a homotopy pullback square

$$\begin{tikzcd} L(A) & L(B) \arrow[l] \\ L(C) \arrow[u] & L(D) \arrow[l] \arrow[u] \end{tikzcd}$$

In particular, L takes finite sums of spaces to finite products of spectra. This type of analysis is similar in spirit to Goodwillie's calculus of functors ([19], [21]), though it is different in substance because the functors F and L are contravariant. It is

perhaps more similar to Weiss's embedding calculus ([36], [22]), though again it is different because F is defined on all spaces and not just manifolds and embeddings.

Of course, in Goodwillie calculus one approximates a functor F by an n-excisive functor P_nF for each integer $n \geq 0$. These fit into a tower

$$F \longrightarrow \ldots \longrightarrow P_nF \longrightarrow \ldots \longrightarrow P_2F \longrightarrow P_1F \longrightarrow P_0F$$

and one extracts information about F from the layers

$$D_nF := \operatorname{hofib}(P_nF \longrightarrow P_{n-1}F)$$

The map of functors (3) described by Cohen and Jones is only the first level of this tower:

$$F \longrightarrow P_1F$$

The main goal of this paper is to extend their construction by building the rest of the tower. In order to do this we must also develop a variant of Goodwillie calculus for contravariant functors from spaces to spectra.

In Definition 2.1 we define n-excisive contravariant functors. Our main results on n-excisive functors are Theorems 7.1 and 8.8, which imply

Theorem 1.1. *Let F be a contravariant homotopy functor from $\mathcal{C}$ to $\mathcal{D}$, where one of the following holds:*

- *$\mathcal{C}$ is the category of unbased finite CW complexes over M, and $\mathcal{D}$ is the category of based spaces or spectra.*
- *$\mathcal{C}$ is the category of based finite CW complexes, and $\mathcal{D}$ is the category of based spaces or spectra.*
- *$\mathcal{C}$ is the category of finite retractive CW complexes over M, and $\mathcal{D}$ is the category of spectra.*

Then there exists a universal n-excisive approximation to F, called P_nF, and the natural transformation $F(X) \longrightarrow P_nF(X)$ is an equivalence when X is a disjoint union of the initial object and i discrete points, $0 \leq i \leq n$.

In section 3 we explicitly define the functors of Cohen and Jones that extend the map of ring spectra (2), and in section 4.2 we explicitly calculate the tower that extends the map of Cohen and Jones. We explain in Proposition 2.5 why the above universal theorem is needed to conclude that our tower is correct. Along the way to proving Theorems 7.1 and 8.8, we prove a splitting result on homotopy limits in Proposition 6.8 that is reminiscient of a result of Dwyer and Kan ([17]) on mapping spaces of diagrams. This all implies the main result of the paper:

Definition. Let $C(M;n)$ denote the space of unordered configurations of n points in M. Let $\mathcal{C}(LM;n)$ denote the space of unordered collections of n free loops in M with distinct basepoints.

Theorem 1.2. *There is a tower of homotopy functors*

$$F \longrightarrow \dots \longrightarrow P_nF \longrightarrow \dots \longrightarrow P_2F \longrightarrow P_1F \longrightarrow P_0F$$

from finite retractive CW complexes over M into spectra such that

(1) *P_nF is the universal n-excisive approximation of F.*
(2) *The map $F \longrightarrow P_1F$ is the map (3) of Cohen and Jones.*
(3) *$F(M \amalg M) \cong \Sigma^\infty_+ \Omega_{\mathrm{id}} \mathrm{haut} M$.*
(4) *$P_1F(M \amalg M) \simeq LM^{-TM}$.*
(5) *$P_0F(M \amalg M) = *$.*
(6) *If X is any finite retractive CW complex over M, the maps*

$$F(X) \longrightarrow P_nF(X) \longrightarrow P_{n-1}F(X)$$

are maps of ring spectra.
(7) *For all $n \geq 1$, $D_nF(M \amalg M)$ is equivalent to the Thom spectrum*

$$\mathcal{C}(LM; n)^{-TC(M;n)}$$

Cohen and Jones have also observed that this linearization phenomenon is more general. Consider any principal bundle

$$G \longrightarrow \mathcal{P} \longrightarrow M$$

The *gauge group* $\mathcal{G}(\mathcal{P})$ is defined to be the space of automorphisms of $\mathcal{P}$ as a principal bundle. It is a classical fact that there is an associated *adjoint bundle* $\mathcal{P}^{\mathrm{Ad}}$, and that the gauge group $\mathcal{G}(\mathcal{P})$ may be identified with the the space of sections $\Gamma_M(\mathcal{P}^{\mathrm{Ad}})$.

Gruher and Salvatore show in [23] that one may construct a Thom ring spectrum $(\mathcal{P}^{\mathrm{Ad}})^{-TM}$ out of the total space $\mathcal{P}^{\mathrm{Ad}}$ of the adjoint bundle. The multiplication on this ring spectrum gives a product on the homology $H_*(\mathcal{P}^{\mathrm{Ad}})$. When the total space of $\mathcal{P}$ is contractible, the adjoint bundle $\mathcal{P}^{\mathrm{Ad}}$ is equivalent to the free loop space LM, and the Gruher-Salvatore product on $H_*(\mathcal{P}^{\mathrm{Ad}})$ agrees with the Chas-Sullivan loop product on $H_*(LM)$.

In [12], Cohen and Jones show that the map (2) of ring spectra generalizes to a map of ring spectra

$$\Sigma^\infty_+ \mathcal{G}(\mathcal{P}) \longrightarrow (\mathcal{P}^{\mathrm{Ad}})^{-TM} \tag{4}$$

Taking homology groups, they get a multiplication-preserving map

$$H_*(\mathcal{G}(\mathcal{P})) \longrightarrow H_{*+\dim M}(\mathcal{P}^{\mathrm{Ad}}) \tag{5}$$

which generalizes the map (1) studied by Félix and Thomas. Cohen and Jones extend this generalized map of ring spectra to a map of functors $F \longrightarrow L$ and show that L is the universal approximation of F by an excisive functor. We extend their generalized result here as well:

Theorem 1.3. *There is a tower of homotopy functors*

$$F \longrightarrow \ldots \longrightarrow P_nF \longrightarrow \ldots \longrightarrow P_2F \longrightarrow P_1F \longrightarrow P_0F$$

from finite retractive CW complexes over M into spectra such that

(1) *P_nF is the universal n-excisive approximation of F.*
(2) *The map $F \longrightarrow P_1F$ is the generalized map of Cohen and Jones.*
(3) *$F(M \amalg M) \cong \Sigma^\infty_+ \mathcal{G}(\mathcal{P})$.*
(4) *$P_1F(M \amalg M) \simeq (\mathcal{P}^{\mathrm{Ad}})^{-TM}$.*
(5) *$P_0F(M \amalg M) = *$.*
(6) *If X is any finite retractive CW complex over M, the maps*

$$F(X) \longrightarrow P_nF(X) \longrightarrow P_{n-1}F(X)$$

are maps of ring spectra.

(7) *For all $n \geq 1$, $D_nF(M \amalg M)$ is equivalent to the Thom spectrum*

$$\mathcal{C}(\mathcal{P}^{\mathrm{Ad}}; n)^{-TC(M;n)}$$

where $\mathcal{C}(\mathcal{P}^{\mathrm{Ad}}; n)$ is the space of unordered configurations of n points in the total space $\mathcal{P}^{\mathrm{Ad}}$ which have distinct images in M.

The outline of the paper is as follows. In Section 2, we define n-excisive functors and give criteria for recognizing the universal n-excisive approximation P_nF of a given functor F. In Section 3, we give a detailed construction of a tower which generalizes the above two examples. In Section 4, we specialize to the above two examples and do some computations. Sections 5-8 supply a missing ingredient from the previous sections: a functorial construction of P_nF for general F. This material may be of independent interest in the general study of calculus of functors.

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2. Excisive Functors

Fix an unbased space B. Like every space that follows, we assume it is compactly generated and weak Hausdorff. Let $\mathcal{U}_B$ be the category of spaces over B, and let $\mathcal{R}_B$ be the category of spaces containing B as a retract. If $B = *$ then these are the familiar categories $\mathcal{U}$ of unbased spaces and $\mathcal{T}$ of based spaces. Additionally, let $\mathcal{U}_{B,\mathrm{fin}}$ and $\mathcal{R}_{B,\mathrm{fin}}$ denote the subcategories of finite CW complexes and finite relative CW complexes.

We will sometimes write X_B as shorthand for $X \amalg B$. So if X is an unbased space, then X_B is a space under B. On the other hand, if $X \longrightarrow B$ is a fiberwise unbased space, then X_B is a retractive space over B.

The following definition should be seen as a contravariant analogue of Goodwillie's notion of n-excisive for covariant functors [20]:

Definition 2.1. A contravariant functor $\mathcal{R}_B^{\mathrm{op}} \xrightarrow{F} \mathcal{T}$ is *n-excisive* if

- F is a *homotopy functor*, meaning weak equivalences $X \xrightarrow{\sim} Y$ of spaces containing B as a retract are sent to weak equivalences $F(Y) \xrightarrow{\sim} F(X)$ of based spaces.
- F takes strongly co-Cartesian cubes of dimension at least $n+1$ to Cartesian cubes (see [20]).
- F takes filtered homotopy colimits to homotopy limits. In particular, F is determined up to equivalence by its behavior on relative finite CW complexes $B \hookrightarrow X$.

This definition is easily modified to suit many cases. When restricting to finite CW complexes $\mathcal{R}_{B,\mathrm{fin}}$, we drop the last condition. If $\mathcal{S}p$ denotes a suitable model category of spectra, for example the category of prespectra described in [27], then a contravariant functor $\mathcal{R}_B^{\mathrm{op}} \xrightarrow{F} \mathcal{S}p$ is *n-excisive* if it satisfies the above properties, with "equivalence of based spaces" replaced by "stable equivalence of spectra." If we post-compose F with a fibrant replacement functor in $\mathcal{S}p$, then F is an n-excisive functor to spectra iff each level F_j is an n-excisive functor to based spaces. It is also straightforward to define *n-excisive* for functors from unbased spaces $\mathcal{U}_B$ or unbased finite spaces $\mathcal{U}_{B,\mathrm{fin}}$ to either spaces $\mathcal{T}$ or spectra $\mathcal{S}p$.

Our goal is to define "best possible" approximations of homotopy functors F by n-excisive functors P. Specifically, we want to construct an n-excisive functor P_nF with the same source and target as F, and a natural transformation $F \longrightarrow P_nF$ that is universal among all maps $F \longrightarrow P$ from F into an n-excisive functor P:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & P \\ P_nF \arrow[ur, dashed, "!"'] & \end{tikzcd}$$

We relax this condition to take place in the *homotopy category* of functors. This is the category we get by formally inverting the equivalences of functors.

Definition 2.2. An *equivalence* of functors is a natural transformation $F \longrightarrow G$ that yields equivalences $F(X) \xrightarrow{\sim} G(X)$ for all spaces X.

Unfortunately, this homotopy category of functors has significant set-theory issues. First of all, the category of functors from spaces to spaces is not really a category in the usual sense. The collection of natural transformations from one functor to another forms a proper class. In other words, the category of functors has large hom-sets. The homotopy category of functors has even larger hom-sets [21].

One way of resolving this issue is to restrict to *small* functors as defined in [8]. The small functors form a model category, so their homotopy category has small hom-sets.

We will use a different fix, since we are ultimately interested in a result about compact manifolds. We will restrict our attention to functors defined on finite CW complexes (i.e. $\mathcal{U}_{B,\mathrm{fin}}$ or $\mathcal{R}_{B,\mathrm{fin}}$) instead of all spaces ($\mathcal{U}_B$ or $\mathcal{R}_B$). Finite CW complexes over B can always be embedded into $B \times \mathbb{R}^\infty$, so we can easily make $\mathcal{U}_{B,\mathrm{fin}}$ and $\mathcal{R}_{B,\mathrm{fin}}$ into small categories. Then the category of functors from $\mathcal{U}_{B,\mathrm{fin}}$ or $\mathcal{R}_{B,\mathrm{fin}}$ into spaces or spectra has the projective model structure, as discussed below in section 3.1.

Now that we are on solid footing, let us return to the problem of finding a universal n-excisive approximation P_nF to the homotopy functor F. It turns out that P_nF actually *agrees* with F on the spaces with at most n points. This is similar to embedding calculus ([36], [22]) but quite different from the case for covariant functors ([21]). Extending the calculus analogy, we are calculating not a Taylor series but a polynomial interpolation: we sample our functor F at $(n+1)$ particular homotopy types $\underline{0}, \ldots, \underline{n}$ and then we build the unique degree n polynomial P_nF that has the same values on those $(n+1)$ homotopy types. The next result says that n-excisive functors are completely determined by their values on these spaces with at most n points.

Definition 2.3.
- Let $\mathbf{O}_{B,n}$ be the full subcategory of $\mathcal{U}_B$ containing one object $\underline{i} = \{1, \ldots, i\}$ for each integer $0 \leq i \leq n$ and each map $\underline{i} \longrightarrow B$. The O stands for "over" since the objects are spaces over B.
- Let $\mathbf{R}_{B,n}$ be the full subcategory of $\mathcal{R}_B$ containing one retractive space of the form $\underline{i}_B := \underline{i} \amalg B$ for each integer $0 \leq i \leq n$ and each map $\underline{i} \longrightarrow B$. The R stands for "retractive" since these spaces contain B as a retract.

Proposition 2.4.
- *If F and G are n-excisive functors $\mathcal{R}_B^{\mathrm{op}} \longrightarrow \mathcal{T}$, and $F \xrightarrow{\eta} G$ is an equivalence on $\mathbf{R}_{B,n}^{\mathrm{op}}$, then η is also an equivalence on $\mathcal{R}_B^{\mathrm{op}}$.*
- *If $F, G : \mathcal{U}_B^{\mathrm{op}} \longrightarrow \mathcal{T}$ and $\eta : F \longrightarrow G$ is an equivalence on $\mathbf{O}_{B,n}^{\mathrm{op}}$ then it is an equivalence on all of $\mathcal{U}_B^{\mathrm{op}}$.*
- *The obvious analogues hold when the source is finite CW complexes $\mathcal{U}_{B,\mathrm{fin}}$ or $\mathcal{R}_{B,\mathrm{fin}}$, or when the target is spectra $\mathcal{S}p$ instead of spaces $\mathcal{T}$.*

Proof. We will prove only the first statement, by induction on the dimension of the relative CW complex $B \longrightarrow X$. The key fact is that a map of Cartesian cubes is an equivalence on the initial vertex if it is an equivalence on all the others.

First we show that η is an equivalence on all spaces of the form $X = \underline{k}_B = \underline{k} \amalg B$, by induction on the integer k. If it is true for $0 \leq k \leq m$ and $m \geq n$, then we can take a partition of $m+1$ into $n+1$ positive integers. This gives a way of expressing $\underline{m+1}_B$ as the final vertex of a pushout cube in which every other vertex is of the form $\underline{k}_B$ with $k \leq m$. The functors F and G take this cube to a Cartesian cube,

and η forms a commuting map between the two Cartesian cubes. This map is an equivalence on every vertex but the initial one, so the map on the initial vertex $F(\underline{m+1}_B) \xrightarrow{\eta} G(\underline{m+1}_B)$ is an equivalence as well.

Before we move to higher dimensional complexes, we need a definition. For each subset $T \subset \{1, 2, \ldots, n\}$, define the *layer cake space* L_T^d to be the subspace of the closed d-dimensional unit cube $[0,1]^d$ consisting of those points whose final coordinate is in the set

$$\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\} \cup \{t : \lceil nt \rceil \in T\}$$

So $L_{\{1,\ldots,n\}}^d$ is the entire cube, while $L_\emptyset^d$ is homotopy equivalent to $n+1$ copies of D^{p-1} glued along their boundaries. Intuitively, L_T^d consists of all the frosting in a layer cake, together with a selection of layers given by T. If T is a proper subset, then L_T^d is homotopy equivalent rel ∂I^d to ∂I^d with some $(d-1)$-cells attached.

Now we want to show that η is an equivalence on all finite relative complexes (i.e. finitely many cells attached to the basepoint section B). Assume that η is an equivalence on all finite $(d-1)$-dimensional complexes. Given a d-dimensional finite complex X, with top-dimensional attaching maps $\{\partial I^d \xrightarrow{\varphi_\alpha} X^{(d-1)}\}_{\alpha \in A}$, form the unique pushout square with the following description. The initial vertex is $\coprod_A L_\emptyset^d$, a disjoint union of one empty layer cake for each d-cell of X. Next, let n of the $n+1$ adjacent vertices be $\coprod_A L_{\{i\}}^d$ as i ranges over $\{1, \ldots, n\}$. Finally, let the last adjacent vertex be the pushout of $X^{(d-1)}$ and $\coprod_A L_\emptyset^d$ along $\coprod_A \partial I^d$. Then the final vertex is homeomorphic to X, while every vertex other than the final one is homotopy equivalent to a $(d-1)$-dimenional cell complex. After applying F and G, η gives us a map between two Cartesian squares, and the map is an equivalence on every vertex but the initial one. So $F(X) \xrightarrow{\eta} G(X)$ is an equivalence and the induction is complete.

If the source category of F and G has infinite CW complexes, we express each CW complex as a filtered homotopy colimit of its finite subcomplexes and invoke the colimit axiom. Then η is an equivalence on all CW complexes. To move to all spaces, we recall that F and G preserve weak equivalences, and that every space over B has a functorial CW approximation. $\square$

Let F be a contravariant homotopy functor from finite CW complexes ($\mathcal{R}_{B,\text{fin}}$ or $\mathcal{U}_{B,\text{fin}}$) to either based spaces or spectra (with one exception, as explained in section 7). In sections 5, 7, and 8.2 below we will define a functor P_nF with the same source and target as F, and a natural transformation $p_nF : F \longrightarrow P_nF$, both functorial in F. Then we will show two things:

- P_nF is n-excisive.
- $F \longrightarrow P_nF$ is an equivalence on $\mathbf{R}_{B,n}^{\text{op}}$ (based case) or $\mathbf{O}_{B,n}^{\text{op}}$ (unbased case).

Proposition 2.5. *If $F \longrightarrow P_n F$ is a functorial construction satisfying the above properties, then $P_n F$ is universal among all n-excisive P with natural transformations $F \longrightarrow P$ in the homotopy category of functors.*

Proof. Easy adaptation of ([21], 1.8). □

Corollary 2.6 (Recognition Principle for $P_n F$)**.** *Given that such a construction P_n exists, if P is any n-excisive functor with a map $F \longrightarrow P$ that is an equivalence on $\mathbf{R}^{\mathrm{op}}_{B,n}$ or $\mathbf{O}^{\mathrm{op}}_{B,n}$, then P is canonically equivalent to $P_n F$.*

Proof. By the universal property of $P_n F$ there exists a unique map $P_n F \longrightarrow P$, but this is a map of n-excisive functors and an equivalence on $\mathbf{R}^{\mathrm{op}}_{B,n}$ or $\mathbf{O}^{\mathrm{op}}_{B,n}$, so it's an equivalence of functors. □

As stated above, we will defer the construction of $P_n F$ to a later section. In the next section we will recognize $P_n F$ in a particular example.

3. The Tower of Approximations of a Mapping Space

Now we will compute the tower of universal n-excisive approximations of the functor

$$F(X) = \Sigma^\infty \mathrm{Map}_B(X, E)$$

from retractive spaces over B to spectra. The map of Cohen and Jones described in the introduction is the special case $X = M \amalg M$, $B = M$, and $E = LM \amalg M$. We will examine this special case in section 4.2 after doing the general case. Most of the properties are proven using techniques from model categories, so we will fix some notation for this following [27] and [29].

Definition 3.1. Let X and Y be unbased spaces over B, or retractive spaces over B.

- A q-cofibration $X \longrightarrow Y$ is a retract of a relative cell complex.
- A q-fibration $X \longrightarrow Y$ is a Serre fibration.
- An h-cofibration $X \longrightarrow Y$ is a map of spaces satisfying the homotopy extension property.
- An h-fibration $X \longrightarrow Y$ is a Hurewicz fibration.
- If E is a retractive space over B, let $\Sigma_B E$ denote the *fiberwise reduced suspension* of E.
- An *ex-fibration* is a retractive space E over B such that $E \longrightarrow B$ is a Hurewicz fibration and $B \longrightarrow E$ has nice properties ([29], 8.2). The point for us is that $\Sigma_B E$ is again an ex-fibration, and fiberwise maps from compact spaces into E will have nondegenerate basepoints.

Definition 3.2. Let X be a q-cofibrant retractive space over B and let E be an ex-fibration over B.

- Let $\mathrm{Map}_B(X, E)$ denote the space of maps $X \longrightarrow E$ respecting the maps into and out of B. If B is compact or X is finite CW then this space is well-based. If not, grow a whisker so that Σ^∞ will preserve equivalences.
- Similarly, let $\mathrm{Map}_B(X, \Sigma^\infty_B E)$ denote a spectrum whose kth level is fiberwise maps from X into $\Sigma^k_B E$.
- Let $X \,\overline{\wedge}\, X$ denote the *external smash product* of X with itself; this is a retractive space over $B \times B$ whose fiber over (b_1, b_2) is $X_{b_1} \wedge X_{b_2}$. More generally, if Y is a retractive space over C then $X \,\overline{\wedge}\, Y$ is a retractive space over $B \times C$ whose fiber over (b, c) is $X_b \wedge Y_c$.
- Let $X^{\overline{\wedge}\, n}$ denote the n-fold iterated external smash product. It is a retractive space over B^n.
- Define

$$\mathrm{Map}_{(\mathbf{M}_n^{\mathrm{op}}, \{B^i\})}(X^{\overline{\wedge}\, i}, \Sigma^\infty_{B^i} E^{\overline{\wedge}\, i})$$

to be the spectrum whose kth level is collections of maps of retractive spaces

$$\begin{tikzcd}
\Sigma^k * & \Sigma^k_B E & \dots & \Sigma^k_{B^n} E^{\overline{\wedge}\, n} \\
* \arrow[u] & X \arrow[u] & \dots & X^{\overline{\wedge}\, n} \arrow[u]
\end{tikzcd}$$

such that each surjective map $\underline{i} \longleftarrow \underline{j}$ gives a commuting square

$$\begin{tikzcd}
\Sigma^k_{B^i} E^{\overline{\wedge}\, i} \arrow[r] & \Sigma^k_{B^j} E^{\overline{\wedge}\, j} \\
X^{\overline{\wedge}\, i} \arrow[u] \arrow[r] & X^{\overline{\wedge}\, j} \arrow[u]
\end{tikzcd}$$

One might expect $S^0 \longrightarrow \Sigma^k S^0$ in the place of $* \longrightarrow \Sigma^k *$, since an empty smash product is S^0. This answer would give the approximation to the functor $F \vee \mathbb{S}$ instead of F. A similar phenomenon happens in Cor. 8.6 below.
- Alternatively, we can define diagrams $\underline{X^{\overline{\wedge}}}$, $\underline{\Sigma^k E}$, and $\underline{B}$ indexed by $\mathbf{M}_n^{\mathrm{op}}$. The first two diagrams each contain the third as a retract. Then we can describe the kth level of the above spectrum as the space of maps of diagrams from $\underline{X^{\overline{\wedge}}}$ to $\underline{\Sigma^k E}$ respecting the structure maps into and out of $\underline{B}$. We will discuss the projective model structure below and show that $\underline{X^{\overline{\wedge}}}$ is cofibrant in Corollary 3.5. Of course, $\underline{\Sigma^k E}$ is fibrant.

This gives a tower of functors

$$
\begin{array}{ccl}
F(X) & = & \Sigma^\infty \mathrm{Map}_B(X, E) \\
\downarrow & & \\
\vdots & & \vdots \\
P_n F(X) & = & \mathrm{Map}_{(\mathbf{M}_n^{\mathrm{op}}, \{B^i\})}(X^{\overline{\wedge} i}, \Sigma^\infty_{B^i} E^{\overline{\wedge} i}) \\
\vdots & & \vdots \\
\downarrow & & \\
P_2 F(X) & = & \mathrm{Map}_{B \times B}(X \overline{\wedge} X, \Sigma^\infty_{B \times B} E \overline{\wedge} E)^{\Sigma_2} \\
\downarrow & & \\
P_1 F(X) & = & \mathrm{Map}_B(X, \Sigma^\infty_B E) \\
\downarrow & & \\
P_0 F(X) & = & *
\end{array}
$$

on the category of finite retractive CW complexes over B. We will justify the notation in Theorem 3.12, where we show that $P_n F(X)$ is the universal n-excisive approximation to $F(X)$. This generalizes an observation made by Greg Arone about the tower he computes in [1]. These functors extend to homotopy functors on all retractive spaces using CW approximation or q-cofibrant approximation, though we will not need that here.

Remark. It is more natural to examine the functor $X \mapsto \mathrm{Map}_B(X, E)$ first, before applying Σ^∞ to it. But this functor is already 1-excisive, so it does not give an interesting tower. It is also natural to consider

$$\hat{F}(X) = \Sigma^\infty \mathrm{Map}_B(X, E)$$

for unbased X over B, without a basepoint section. But $\hat{F}(X) = F(X \amalg B)$, so $P_n \hat{F}(X) = P_n F(X \amalg B)$ by comparing universal properties. Therefore the case of F on $\mathcal{R}_B^{\mathrm{op}}$ is more general than the case of $\hat{F}$ on $\mathcal{U}_B^{\mathrm{op}}$. This is true in general when the desired functor on $\mathcal{U}_B^{\mathrm{op}}$ extends to a functor on $\mathcal{R}_B^{\mathrm{op}}$.

3.1. **Cell Complexes of Diagrams.** Many of the proofs that follow rely on the same basic idea: we start with a diagram of spaces or spectra that is built inductively out of cells, and we define maps of diagrams one cell at a time. In doing so, we are using the following standard facts. First, both spaces and spectra have compactly generated model structures [27], so each space or spectrum is equivalent to one that is built out of a sequential colimit of pushouts of cells. Therefore the category of diagrams indexed by $\mathbf{I}$ can be endowed with the *projective model structure*. The weak equivalences $F \longrightarrow G$ are the maps that give objectwise equivalences $F(i) \xrightarrow{\sim} G(i)$, and the fibrations are the objectwise (q-)fibrations. The projective model structure is again compactly generated.

To understand the cofibrant diagrams, define a functor that takes a based space (or spectrum) X and produces the diagram

$$F_i(X)(j) = \mathbf{I}(i,j)_+ \wedge X$$

A map of diagrams $F_i(X) \longrightarrow G$ is the same as a map of spaces (or spectra) $X \longrightarrow G(i)$, a property that is clearly useful for defining maps of diagrams one cell at a time. We can define a *diagram cell* by applying F_i to the maps $S^{n-1}_+ \longrightarrow D^n_+$, and then define a *diagram CW complex* as an appropriate iterated pushout of diagram cells. Every diagram CW complex is cofibrant in the projective model structure. More generally, if we weaken the definition from relative CW complexes to retracts of relative cell complexes, we get all of the cofibrations in the projective model structure.

We will check that an important class of diagrams is cofibrant in the projective model structure. Recall that $\mathbf{M}_n$ is the category with one object $\underline{i} = \{1,\ldots,i\}$ for each integer $0 \leq i \leq n$, with maps $\underline{i} \longrightarrow \underline{j}$ the surjective maps of sets. The maps are not required to preserve ordering, so in particular $\mathbf{M}_n(\underline{i},\underline{i}) \cong \Sigma_i$, the symmetric group on i letters.

Proposition 3.3. *If X is a based CW complex, then $\{X^{\wedge i}\}_{i=0}^n$ is a CW complex of $\mathbf{M}_n^{\mathrm{op}}$ diagrams. Similarly for Cartesian products $\{X^i\}$. If X is q-cofibrant then $\{X^{\wedge i}\}$ and $\{X^i\}$ are cofibrant diagrams.*

Proof. Consider the product $\prod^k D^m \cong \prod^k [0,1]^m$, which is homeomorphic to the space of all $k \times m$ matrices, with real entries between 0 and 1. There is a Σ_k action that permutes the rows. Divide this space into open simplices as follows. We define a simplex of dimension d for each partition of the km entries of the matrix into d nonempty equivalence classes, along with a choice of total ordering on the equivalence classes. This simplex corresponds to the subspace of matrices for which the equivalent entries have the same value, and the values are ordered according to the chosen total ordering.

The closures of these simplices give a triangulation of the cube $\prod^k D^m \cong \prod^k [0,1]^m$. Each generalized diagonal in $(D^m)^k$ is defined by setting an equivalence relation on the rows of the matrix, and requiring that equivalent rows have the same values. This is clearly an intersection of conditions we used to define the simplices above, so each generalized diagonal is a union of simplices. In addition, the simplices off the fat diagonal are freely permuted by the Σ_k action.

Now, each cell of $X^{\wedge n}$ is an n-tuple of non-basepoint cells $D^{m_1} \times \ldots \times D^{m_n}$ from X. For each cell D^m of X, it occurs k times in this product, with $0 \leq k \leq n$. Subdivide $\prod^k D^m$ as above, and then take an open cell in $\prod_{i=1}^n D^{m_i}$ to be a product of the open cells we constructed that way. Again, each generalized diagonal is a union of cells, since the condition that certain coordinates agree means that certain D^ms must be repeated, and then that condition descends to $\prod^k D^m$, where we have a smaller generalized diagonal. Again, off the fat diagonal, each point is contained in

some cell off the fat diagonal in each of $\prod^k D^m$, each of which is permuted freely by Σ_k, so their product is permuted freely by Σ_n.

This breaks up each basic cell $\prod_{i=1}^n D^{m_i}$ of $X^{\wedge n}$ into a relative CW complex with the desired properties. This process gives the same answer for $X^{\wedge(n-1)}$ as it does when we do it to $X^{\wedge n}$ and then restrict to a generalized diagonal. Inductively, the diagram $\{X^{\wedge i}\}_{i=0}^n$ is therefore built out of free $\mathbf{M}_n^{\mathrm{op}}$-cells, so it is cofibrant.

If X is merely a cell complex, apply the above argument to each relative cell, and simply drop the dimension requirements. The result is a cell complex $X^{\wedge n}$ with the same properties off the fat diagonal. A general q-cofibration is a retract of a relative cell complex, but retracts of maps of spaces clearly give retracts of maps of diagrams, so we are done. $\square$

Proposition 3.4. *If X is a based CW complex and A is a subcomplex then $\{A^{\wedge i}\} \longrightarrow \{X^{\wedge i}\}$ is a relative CW complex of $\mathbf{M}_n^{\mathrm{op}}$ diagrams. If $* \longrightarrow A \longrightarrow X$ are q-cofibrations then $\{A^{\wedge i}\} \longrightarrow \{X^{\wedge i}\}$ is a cofibration of $\mathbf{M}_n^{\mathrm{op}}$ diagrams.*

Proof. Each cell of $X^{\wedge i}$ lying outside $A^{\wedge i}$ is a product of cells in X, at least one of which is not a cell in A. As above, we subdivide each of these cells so that $\Delta \cup A^{\wedge i}$ is a subcomplex when Δ is any of the generalized diagonals. Off the fat diagonal, the Σ_i action still freely permutes the cells. This gives the recipe for building the map of diagrams $\{A^{\wedge i}\} \longrightarrow \{X^{\wedge i}\}$ out of free cells of diagrams.

Suppose that $* \longrightarrow A \longrightarrow X$ are q-cofibrations, and we want to show that $\{A^{\wedge i}\} \longrightarrow \{X^{\wedge i}\}$ is a cofibration of diagrams. Then without loss of generality we can replace $A \longrightarrow X$ by a relative cell complex $A \longrightarrow X'$. Then we can replace $* \longrightarrow A$ by a relative cell complex $* \longrightarrow A'$, and we get the sequence of relative cell complexes $* \longrightarrow A' \longrightarrow X' \cup_A A'$ containing $* \longrightarrow A \longrightarrow X$ as a retract. Then we apply the same argument as above. $\square$

Proposition 3.5. *If X is a retractive CW complex over B then $\{B^i\} \longrightarrow \{X^{\overline{\wedge} i}\}$ is a relative CW complex of $\mathbf{M}_n^{\mathrm{op}}$ diagrams. If $B \longrightarrow A \longrightarrow X$ are q-cofibrations over B then $\{A^{\overline{\wedge} i}\} \longrightarrow \{X^{\overline{\wedge} i}\}$ is a cofibration of $\mathbf{M}_n^{\mathrm{op}}$ diagrams.*

Proof. We must verify that $B^i \hookrightarrow X^{\overline{\wedge} i}$ is a relative cell complex with one cell for each i-tuple of relative cells of $B \hookrightarrow X$. This is a straightforward adaptation of standard arguments, though it is worth pointing out that these arguments derail if we don't work in the category of compactly generated weak Hausdorff spaces. Once we are assured that everything is a cell complex, the rest of the proof follows as above. $\square$

Recall that an *acyclic cofibration* is a map that is both a (q-)cofibration and a weak equivalence. An *acyclic cell* of spaces is a map $D^n \times \{0\} \hookrightarrow D^n \times I$ for some $n \geq 0$. Every acyclic cofibration of spaces is a retract of a cell complex built out of these acyclic cells [27]. Similarly, an acyclic cell of diagrams is what we get by applying

F_i to the map $D^n \times \{0\} \hookrightarrow D^n \times I$, and every acyclic cofibration of diagrams is a retract of a relative complex built out of these acyclic cells. With this language, we now give the following result:

Corollary 3.6. *If $* \longrightarrow A \longrightarrow X$ are q-cofibrations and $A \longrightarrow X$ is acyclic then $\{A^{\wedge i}\} \longrightarrow \{X^{\wedge i}\}$ is an acyclic cofibration of $\mathbf{M}_n^{\mathrm{op}}$ diagrams. Similarly for Cartesian products.*

Proof. Since A and X are q-cofibrant they are well-based (i.e. $* \longrightarrow A$ is an h-cofibration). Therefore since $A \longrightarrow X$ is a weak equivalence, $A^{\wedge i} \longrightarrow X^{\wedge i}$ is a weak equivalence as well. $\square$

Corollary 3.7. *Each acyclic cofibration $A \longrightarrow X$ of q-cofibrant retractive spaces over B induces a acyclic cofibration of $\mathbf{M}_n^{\mathrm{op}}$ diagrams $\{A^{\overline{\wedge} i}\} \longrightarrow \{X^{\overline{\wedge} i}\}$.*

Proof. Again, we just need to show that $A^{\overline{\wedge} i} \longrightarrow X^{\overline{\wedge} i}$ is a weak equivalence of total spaces. The case where $i = 2$ generalizes easily. Let H_A be the homotopy pushout of

$$\begin{tikzcd} B \times B & \\ (A \times B) \cup_{B \times B} (B \times A) \arrow[u] \arrow[r] & A \times A \end{tikzcd}$$

Then H_A is equivalent to the strict pushout $A \overline{\wedge} A$, because the bottom map is an h-cofibration. This gives a square

$$\begin{tikzcd} H_A \arrow[r, "\sim"] \arrow[d, "\sim"] & H_X \arrow[d, "\sim"] \\ A \overline{\wedge} A \arrow[r] & X \overline{\wedge} X \end{tikzcd}$$

from which we see that the bottom map is an equivalence. For $i > 2$ simply replace one of the two copies of A by the space $A^{\overline{\wedge}(i-1)}$. $\square$

3.2. **Proof that the Tower is Correct.**

Proposition 3.8. *$F(X) = \Sigma^\infty \mathrm{Map}_B(X, E)$ as defined above in 3.2 takes weak equivalences between q-cofibrant retractive spaces over B to level equivalences of spectra. In particular, F is a homotopy functor on the relative CW complexes over B.*

Proof. Since the spaces are modified to be well-based, it suffices to do this for the functor $\mathrm{Map}_B(X, E)$. By Ken Brown's lemma, it suffices to take an acyclic q-cofibration $X \longrightarrow Y$ and show that

$$\mathrm{Map}_B(Y, E) \longrightarrow \mathrm{Map}_B(X, E)$$

is a weak equivalence. So take the square

$$\begin{tikzcd} S^{n-1}_+ \arrow[r] \arrow[d] & \mathrm{Map}_B(Y, E) \arrow[d] \\ D^n_+ \arrow[r] \arrow[ur, dashed] & \mathrm{Map}_B(X, E) \end{tikzcd}$$

where the + means disjoint basepoint and is there to remind us that the map must be an isomorphism on homotopy groups at all points. The right-hand vertical map is a weak equivalence (actually an acyclic fibration) if we can show the dotted diagonal map exists. This is equivalent to

$$\begin{tikzcd} (S^{n-1} \times Y) \cup (D^n \times X) \arrow[r] \arrow[d] & E \arrow[d] \\ D^n \times Y \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

Since $E \longrightarrow B$ is a q-fibration, it suffices to show that the left-hand vertical map is an acyclic q-cofibration. This is the main axiom for checking that (compactly generated) spaces form a monoidal model category, and it follows easily by checking a similar condition on the generating maps $S^{n-1} \longrightarrow D^n$ and $D^n \times \{0\} \longrightarrow D^n \times I$ [25]. Note that the homotopy invariance of mapping spaces from cofibrant objects to fibrant objects could also be deduced from the results of Dwyer and Kan on hammock localization [17]. $\square$

Proposition 3.9. $P_n F(X) = \mathrm{Map}_{(\mathbf{M}_n^{\mathrm{op}}, \{B^i\})}(X^{\barwedge i}, \Sigma^\infty_{B^i} E^{\barwedge i})$ *as defined above in 3.2 takes weak equivalences between q-cofibrant retractive spaces over B to level equivalences of spectra.*

Proof. Again, by Ken Brown's lemma it suffices to take an acyclic q-cofibration $X \longrightarrow Y$ and show that

$$\mathrm{Map}_{(\mathbf{M}_n^{\mathrm{op}}, \{B^i\})}(Y^{\barwedge i}, \Sigma^\infty_{B^i} E^{\barwedge i}) \longrightarrow \mathrm{Map}_{(\mathbf{M}_n^{\mathrm{op}}, \{B^i\})}(X^{\barwedge i}, \Sigma^\infty_{B^i} E^{\barwedge i})$$

is a level equivalence of spectra. So take the square of spaces

$$\begin{tikzcd} S^{n-1}_+ \arrow[r] \arrow[d] & \mathrm{Map}_{(\mathbf{M}_n^{\mathrm{op}}, \{B^i\})}(Y^{\barwedge i}, \Sigma^k_{B^i} E^{\barwedge i}) \arrow[d] \\ D^n_+ \arrow[r] \arrow[ur, dashed] & \mathrm{Map}_{(\mathbf{M}_n^{\mathrm{op}}, \{B^i\})}(X^{\barwedge i}, \Sigma^k_{B^i} E^{\barwedge i}) \end{tikzcd}$$

and show the dotted diagonal map exists. This is equivalent to a lift in this square of diagrams indexed by $\mathbf{M}_n^{\mathrm{op}}$:

$$\begin{tikzcd} (S^{n-1}_+ \barwedge Y^{\barwedge i}) \cup (D^n_+ \barwedge X^{\barwedge i}) \arrow[r] \arrow[d] & \Sigma^k_{B^i} E^{\barwedge i} \arrow[d] \\ D^n_+ \barwedge Y^{\barwedge i} \arrow[r] \arrow[ur, dashed] & B^i \end{tikzcd}$$

Again, the right-hand vertical map is an ex-fibration ([29], 8.2.4) and therefore a q-fibration. So it suffices to show the left-hand vertical map is an acyclic cofibration of diagrams. Using ([29], 7.3.2), this reduces to showing that

$$X^{\overline{\wedge} i} \longrightarrow Y^{\overline{\wedge} i}$$

is an acyclic cofibration of diagrams, but we already did that in Prop. 3.7 above. □

Proposition 3.10. *$F \longrightarrow P_n F$ is an equivalence on the q-cofibrant space $\underline{i}_B = \underline{i} \amalg B$ when $0 \le i \le n$.*

Proof. Note that when $X = \underline{i} \amalg B$, the fat diagonal covers all of $X^{\overline{\wedge} j}$ for $j > i$. Therefore a natural transformation of $\mathbf{M}_n^{\mathrm{op}}$-diagrams is determined by what it does on $X^{\overline{\wedge} i} = \underline{i}^i \amalg B$. This is an i^i-tuple of points in various fibers of $\Sigma^\infty_{B^i} E^{\overline{\wedge} i}$, with compatibility conditions. The compatibility conditions force us to have only one point for each nonempty subset of $\underline{i}$. Therefore the map $F(\underline{i}_B) \longrightarrow P_n F(\underline{i}_B)$ becomes

$$\Sigma^\infty (E_{b_1} \times \ldots \times E_{b_i}) \longrightarrow \prod_{S \subset \underline{i}, S \neq \emptyset} \Sigma^\infty \bigwedge_{s \in S} E_{b_s}$$

From Cor. 8.6 below, this map is always an equivalence. □

Proposition 3.11. *$P_n F$ is n-excisive.*

Proof. Start with a strongly co-Cartesian cube indexed by the subsets of a fixed finite set S, with $|S| \ge n+1$. This cube is equivalent to a cube of pushouts along relative CW complexes

$$A \longrightarrow X_s \qquad s \in S$$

of retractive spaces over B. Applying $P_n F$ to this cube, we get a cube of spectra

$$T \rightsquigarrow P_n F(\bigcup_{s \in T} X_s)$$

Let's show that this cube is level Cartesian. Fix a nonnegative integer k and restrict attention to level k of the spectra in the cube. This turns out to be a *fibration cube* as defined in [20]. To prove this, we have to construct this lift for any space K:

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & \mathrm{Map}_{(\mathbf{M}_n^{\mathrm{op}}, \{B^i\})}\left(\left(\bigcup_{s \in S} X_s\right)^{\overline{\wedge} i}, \Sigma^k_{B^i} E^{\overline{\wedge} i}\right) \arrow[d] \\
K \times I \arrow[r] \arrow[ur, dashed] & \lim\limits_{T \subsetneq S} \mathrm{Map}_{(\mathbf{M}_n^{\mathrm{op}}, \{B^i\})}\left(\left(\bigcup_{t \in T} X_t\right)^{\overline{\wedge} i}, \Sigma^k_{B^i} E^{\overline{\wedge} i}\right) \arrow[d, equal] \\
 & \mathrm{Map}_{(\mathbf{M}_n^{\mathrm{op}}, \{B^i\})}\left(\operatorname*{colim}_{T \subsetneq S} \left(\bigcup_{t \in T} X_t\right)^{\overline{\wedge} i}, \Sigma^k_{B^i} E^{\overline{\wedge} i}\right)
\end{tikzcd}$$

Rearranging gives

$$\begin{tikzcd}
K \times \left[\left(\{0\} \times \left(\bigcup_{s \in S} X_s\right)^{\overline{\wedge} i}\right) \cup \left(I \times \operatorname*{colim}_{T \subsetneq S} \left(\bigcup_{t \in T} X_t\right)^{\overline{\wedge} i}\right)\right] \arrow[r] \arrow[d] & \Sigma^k_{B^i} E^{\overline{\wedge} i} \arrow[d] \\
K \times I \times \left(\bigcup_{s \in S} X_s\right)^{\overline{\wedge} i} \arrow[r] \arrow[ur, dashed] & B^i
\end{tikzcd}$$

This is a square of maps of diagrams in which the left and right vertical maps are vertexwise h-cofibrant and h-fibrant, respectively. Unfortunately, our model structure on diagrams is q-type, not h-type. Fortunately, we can define maps of $\mathbf{M}_n^{\mathrm{op}}$-diagrams one level at a time, one cell at a time. So consider inductively the modified square

$$\begin{tikzcd}
K \times \left[\left(\{0\} \times \left(\bigcup_{s \in S} X_s\right)^{\overline{\wedge} i}\right) \cup \left(I \times \left[\Delta \cup \operatorname*{colim}_{T \subsetneq S} \left(\bigcup_{t \in T} X_t\right)^{\overline{\wedge} i}\right]\right)\right] \arrow[r] \arrow[d] & \Sigma^k_{B^i} E^{\overline{\wedge} i} \arrow[d] \\
K \times I \times \left(\bigcup_{s \in S} X_s\right)^{\overline{\wedge} i} \arrow[r] \arrow[ur, dashed] & B^i
\end{tikzcd}$$

where $\Delta \subset (\bigcup_{s \in S} X_s)^{\overline{\wedge} i}$ is the fat diagonal. From Prop. 3.5 above we know that $(\bigcup_S X_s)^{\overline{\wedge} i}$ is built up from its fat diagonal by attaching free Σ_i-cells, so we can define the lift one free Σ_i-cell at a time. Each time, we get an acyclic h-cofibration on the left, and the map on the right is an h-fibration, so the lift exists. By construction, it's natural with respect to all the maps in $\mathbf{M}_n^{\mathrm{op}}$.

Now that we have a fibration cube of spaces

$$(T \subset S) \mapsto \operatorname{Map}\left(\left(\bigcup_{t \in T} X_t\right)^{\overline{\wedge} i}, \Sigma^k_{B^i} E^{\overline{\wedge} i}\right)$$

we check that the map from the initial vertex into the ordinary limit of the rest is a weak equivalence:

$$\begin{tikzcd}
\operatorname{Map}_{(\mathbf{M}_n^{\mathrm{op}}, \{B^i\})}\left(\left(\bigcup_{s \in S} X_s\right)^{\overline{\wedge} i}, \Sigma^k_{B^i} E^{\overline{\wedge} i}\right) \arrow[r] & \lim\limits_{T \subsetneq S} \operatorname{Map}_{(\mathbf{M}_n^{\mathrm{op}}, \{B^i\})}\left(\left(\bigcup_{t \in T} X_t\right)^{\overline{\wedge} i}, \Sigma^k_{B^i} E^{\overline{\wedge} i}\right) \arrow[d, equal] \\
& \operatorname{Map}_{(\mathbf{M}_n^{\mathrm{op}}, \{B^i\})}\left(\operatorname*{colim}_{T \subsetneq S} \left[\left(\bigcup_{t \in T} X_t\right)^{\overline{\wedge} i}\right], \Sigma^k_{B^i} E^{\overline{\wedge} i}\right)
\end{tikzcd}$$

But since $i \leq n < |S|$, every choice of i points in $\bigcup_S X_s$ lies in some $\bigcup_T X_t$ for some proper subset T of S. Therefore this map is a homeomorphism. $\square$

Theorem 3.12. *$P_n F$ is the universal n-excisive approximation of F.*

Proof. This follows from Cor. 2.6 above and Thm. 8.8 below. Together, they tell us that the universal n-excisive approximation $P_n F$ exists and is uniquely identified

by the property that P_nF is n-excisive and $F \longrightarrow P_nF$ is an equivalence on the spaces with at most n points. $\square$

3.3. **The Layers.** As before, let $\Delta \subset X^{\overline{\wedge}\, n}$ be the fat diagonal, the space of all n-tuples with a repeated coordinate $(\ldots, x_i, x_i, \ldots) \longrightarrow (\ldots, b_i, b_i, \ldots)$. The natural map $P_nF \longrightarrow P_{n-1}F$ is a level fibration of spectra:

$$\begin{tikzcd}
D^m \arrow[r] \arrow[d] & \mathrm{Map}_{(\mathbf{M}_n^{\mathrm{op}}, \{B^i\})}(X^{\overline{\wedge}\, i}, \Sigma^k_{B^i} E^{\overline{\wedge}\, i}) \arrow[d] \\
D^m \times I \arrow[r] \arrow[ur, dashed] & \mathrm{Map}_{(\mathbf{M}_{n-1}^{\mathrm{op}}, \{B^i\})}(X^{\overline{\wedge}\, i}, \Sigma^k_{B^i} E^{\overline{\wedge}\, i})
\end{tikzcd}$$

This rearranges to

$$\begin{tikzcd}
D^m \times \Big((\{0\} \times X^{\overline{\wedge}\, n}) \cup (I \times \Delta) \Big) \arrow[r] \arrow[d] & \Sigma^k_{B^n} E^{\overline{\wedge}\, n} \arrow[d] \\
D^m \times I \times X^{\overline{\wedge}\, n} \arrow[r] \arrow[ur, dashed] & B^n
\end{tikzcd}$$

and the lift exists because the left vertical is an acyclic q-fibration and the right vertical is an h-fibration, therefore a q-fibration.

Since the maps between levels are Serre fibrations, the ordinary fiber is equivalent to the homotopy fiber. The ordinary fiber consists of all collections of maps that are trivial on $X^{\overline{\wedge}\, i}$ for $i < n$ and that vanish on the fat diagonal of $X^{\overline{\wedge}\, n}$. This spectrum may be written

$$D_nF(X) \simeq \mathrm{Map}_{B^n}(X^{\overline{\wedge}\, n}/_{B^n}\Delta, \Sigma^\infty_{B^n} E^{\overline{\wedge}\, n})^{\Sigma_n}$$

although we should be clear about the meaning of the decoration $(-)^{\Sigma_n}$, since in the world of equivariant spectra there are a few different notions of "fixed points" (see for example [26]). Here we using the most literal notion, in which the fixed points of a spectrum X is given at level k by the fixed points of X_k. In particular, the above spectrum is given at level k by the Σ_n-equivariant maps

$$X^{\overline{\wedge}\, n}/_{B^n}\Delta \longrightarrow \Sigma^k_{B^n} E^{\overline{\wedge}\, n}$$

of retractive spaces over B^n.

Proposition 3.13. *$D_nF(X)$ is an $(m-d)n-1$ connected spectrum, where $d = \dim X$ and m is the connectivity of $E \longrightarrow B$ (and so $m-1$ is the connectivity of the fiber E_b).*

Proof. If Y is a c-connected based space and X is a d-dimensional based free Σ_n-CW complex, then $\mathrm{Map}(X, Y)^{\Sigma_n}$ has connectivity at least $c - d$. We prove this by constructing an equivariant homotopy of $S^k \wedge X \longrightarrow Y$ to the constant map, one free Σ_n-cell at a time. A straightforward adaptation of this argument gives the above result. $\square$

Corollary 3.14. *If $m > d$ then the tower converges to its limit*

$$P_\infty F(X) = \mathrm{Map}_{(\mathbf{M}_\infty^{\mathrm{op}}, \{B^i\})}(X^{\overline{\wedge} i}, \Sigma^\infty_{B^i} E^{\overline{\wedge} i})$$

The tower may converge to $F(X)$ itself when $m > d$. In the case where $B = *$ this is shown to be true in [1]. For general B, here is a partial result:

Proposition 3.15. *If $m > d$ then the map $F \longrightarrow P_0 F$ is $m - d$ connected and the map $F \longrightarrow P_1 F$ is $2(m-d) - 1$ connected.*

Proof. First we replace every Σ^∞ with $Q = \Omega^\infty \Sigma^\infty$; this doesn't change the homotopy groups under the assumption that $m > d$ because our spectra are connective. The first result follows easily from the fact that if X is k connected then so is QX. The second follows from the fact that if X is well-based and k connected then $X \longrightarrow QX$ is $2k+1$ connected. (This in turn comes from the Freudenthal Suspension theorem.) We look at the diagram

$$\begin{tikzcd}
\mathrm{Map}_B(X, E) \arrow[r] \arrow[d] & Q\mathrm{Map}_B(X, E) \arrow[dl] \\
\mathrm{Map}_B(X, Q_B E) &
\end{tikzcd}$$

The vertical map is $2m - d - 1$ connected and the horizontal map is $2m - 2d - 1$ connected, so the diagonal is $2m - 2d - 1$ connected. $\square$

If M is a closed manifold, consider the following spaces:

- $\Delta \subset M^n$ is the fat diagonal.
- $F(M;n) \cong M^n - \Delta$ is the noncompact manifold of ordered n-tuples of distinct points in M.
- $\iota : M^n - \Delta \hookrightarrow M^n$ is the inclusion map.
- $C(M;n) \cong F(M;n)_{\Sigma_n}$ is the noncompact manifold of unordered n-tuples of distinct points in M.

Then when $B = M$ and $X = M \amalg M$ the layers of the above tower can be rewritten

$$\begin{aligned}
D_n F(M \amalg M) &= \Gamma_{(M^n, \Delta)}(\Sigma^\infty_{M^n} E^{\overline{\wedge} n})^{\Sigma_n} \\
&\simeq \Gamma^c_{M^n - \Delta}\left(\Sigma^\infty_{M^n} E^{\overline{\wedge} n}\Big|_{M^n - \Delta}\right)^{\Sigma_n} \\
&\cong \Gamma^c_{F(M;n)}\left(\Sigma^\infty_{F(M;n)} \iota^* E^{\overline{\wedge} n}\right)^{\Sigma_n} \\
&\cong \Gamma^c_{C(M;n)}(\Sigma^\infty_{C(M;n)}(\iota^* E^{\overline{\wedge} n})_{\Sigma_n}) \\
&\simeq ((\iota^* E^{\overline{\wedge} n})_{\Sigma_n})^{-T(C(M;n))}
\end{aligned}$$

The last step is the application of Poincaré duality (see [29], [14]) to the noncompact manifold $C(M;n)$ with twisted coefficients given by the bundle of spectra $(\iota^* E^{\overline{\wedge} n})_{\Sigma_n}$. Since the manifold in question is noncompact, Poincaré duality gives

an equivalence between cohomology with compact supports and homology desuspended by the tangent bundle of $C(M;n)$. The result is the Thom spectrum

$$((\iota^* E^{\overline{\wedge} n})_{\Sigma_n})^{-T(C(M;n))}$$

We will see a few examples of this in the next section.

4. Examples and Calculations

Example 4.1. Taking $B = *$ and $E = Y$ for any based space Y gives

$$\begin{array}{rcl} F(X) & = & \Sigma^\infty \mathrm{Map}_*(X,Y) \\ \vdots & & \vdots \\ P_n F(X) & = & \mathrm{Map}_{\mathbf{M}_n^{\mathrm{op}},*}(X^{\wedge i}, \Sigma^\infty Y^{\wedge i}) \\ \vdots & & \vdots \\ P_2 F(X) & = & \mathrm{Map}_*(X \wedge X, \Sigma^\infty Y \wedge Y)^{\Sigma_2} \\ P_1 F(X) & = & \mathrm{Map}_*(X, \Sigma^\infty Y) \\ P_0 F(X) & = & * \end{array}$$

with nth layer

$$D_n F(X) = \mathrm{Map}_*(X^{\wedge n}, \Sigma^\infty Y^{\wedge n})^{\Sigma_n}$$

This coincides with Arone's tower in [1], and therefore converges when the connectivity of Y is at least the dimension of X. It is curious that the Taylor tower in the Y variable should agree with the polynomial interpolation tower in the X variable. We also expect this to happen in the case $B \neq *$, though we will not prove it here.

Example 4.2. If $X = S^1$ and Y is simply connected then the nth layer of the tower is

$$\mathrm{Map}_*(S^n/\Delta, \Sigma^\infty Y^{\wedge n})^{\Sigma_n} \cong \Omega^n \Sigma^\infty Y^{\wedge n}$$

If $Y = \Sigma Z$ with Z connected, then the nth layer is

$$\Sigma^\infty Z^{\wedge n}$$

It is well known that the tower splits in this case ([1], [2]):

$$\Sigma^\infty \Omega \Sigma Z \simeq \prod_{n=1}^{\infty} \Sigma^\infty Z^{\wedge n}$$

Example 4.3. If X is unbased we get the tower

$$\begin{array}{rcl} F(X) & = & \Sigma^\infty \mathrm{Map}(X, Y) \\ \vdots & & \vdots \\ P_n F(X) & = & \mathrm{Map}_{\mathbf{M}_n^{\mathrm{op}}}(X^i, \Sigma^\infty Y^{\wedge i}) \\ \vdots & & \vdots \\ P_2 F(X) & = & \mathrm{Map}(X \times X, \Sigma^\infty Y \wedge Y)^{\Sigma_2} \\ P_1 F(X) & = & \mathrm{Map}(X, \Sigma^\infty Y) \\ P_0 F(X) & = & * \end{array}$$

If $Y = S^0$ and X is any finite unbased CW complex then the nth layer of this tower is

$$D_n F(X) \simeq \mathrm{Map}(X^n/\Delta, \mathbb{S})^{\Sigma_n} \cong D((X^n/\Delta)_{\Sigma_n}) \cong D(C(X; n))$$

where D denotes Spanier-Whitehead dual. If $Y = S^m$ and $m > \dim X$ then the tower converges to $\Sigma^\infty \mathrm{Map}(X, S^m)$, and the nth layer is

$$D_n F(X) \simeq \mathrm{Map}(X^n/\Delta, \Sigma^{mn}\mathbb{S})^{\Sigma_n} \simeq \Sigma^{mn} D((X^n/\Delta)_{\Sigma_n}) \cong \Sigma^{mn} D(C(X; n))$$

Example 4.4. Returning to the fiberwise case, suppose that E is of the form $\widetilde{E} \amalg B$, where $\widetilde{E}$ also has a chosen section. Then

$$S(X) = \Sigma^\infty \mathrm{Map}_B(X, S^0 \times B) \cong \Sigma^\infty \mathrm{Map}_*(X/B, S^0)$$

is a retract of

$$F(X) = \Sigma^\infty \mathrm{Map}_B(X, E)$$

and therefore the tower for F splits into the tower for S and a "reduced" tower for F. The nth layer of the reduced tower is the homotopy cofiber of $D_n S(X) \longrightarrow D_n F(X)$, or equivalently the homotopy fiber of $D_n F(X) \longrightarrow D_n S(X)$, as in Cor. 8.2 below.

4.1. **Gauge Groups and Thom Spectra.** Let $B = M$ be a closed connected manifold, and let $\mathcal{P} \longrightarrow M$ be a G-principal bundle. The *gauge group* $\mathcal{G}(\mathcal{P})$ is defined to be the space of automorphisms of $\mathcal{P}$ as a principal bundle. Consider the quotient

$$\mathcal{P}^{\mathrm{Ad}} = \mathcal{P} \times_G G^{\mathrm{Ad}}$$

where G^{Ad} is the group G as a right G-space with the conjugation action. Then we may identify $\mathcal{G}(\mathcal{P})$ with the space of sections $\Gamma_M(\mathcal{P}^{\mathrm{Ad}})$. Taking E to be the

ex-fibration $(\mathcal{P}^{\mathrm{Ad}}) \amalg M$ and X to be the retractive space $M \amalg M$ gives the tower

$$\begin{array}{rcccc}
F(M \amalg M) & = & \Sigma^\infty \Gamma_M(\mathcal{P}^{\mathrm{Ad}} \amalg M) & \cong & \Sigma^\infty_+ \mathcal{G}(\mathcal{P}) \\
 & & \vdots & & \\
P_n F(M \amalg M) & = & \Gamma_{(\mathbf{M}_n^{\mathrm{op}}, \{M^i\})}(\Sigma^\infty_{M^i} (\mathcal{P}^{\mathrm{Ad}})^i \amalg M^i) & & \\
 & & \vdots & & \\
P_2 F(M \amalg M) & = & \Gamma_{M \times M}(\Sigma^\infty_{M \times M} (\mathcal{P}^{\mathrm{Ad}})^2 \amalg M^2)^{\Sigma_2} & & \\
P_1 F(M \amalg M) & = & \Gamma_M(\Sigma^\infty_M \mathcal{P}^{\mathrm{Ad}} \amalg M) & \simeq & (\mathcal{P}^{\mathrm{Ad}})^{-TM} \\
P_0 F(M \amalg M) & = & * & &
\end{array}$$

The description of the nth layer in section 3.3 above becomes

$$D_n F(M \amalg M) \simeq \mathcal{C}(\mathcal{P}^{\mathrm{Ad}}; n)^{-T(C(M;n))}$$

Here $\mathcal{C}(\mathcal{P}^{\mathrm{Ad}}; n)$ is configurations of n points in $\mathcal{P}^{\mathrm{Ad}}$ with distinct images in $C(M; n)$. If we use orthogonal spectra instead of prespectra, we get a tower of strictly associative ring spectra. This proves Theorem 1.3 from the introduction. If G is replaced by a grouplike A_∞ space then we get a tower of A_∞ ring spectra.

The bundle $\mathcal{P}^{\mathrm{Ad}}$ contains a canonical section, corresponding to the identity map of $\mathcal{P}$. Therefore there is a reduced version of the above tower. The top of the reduced tower is

$$\Sigma^\infty \Gamma_M(\mathcal{P}^{\mathrm{Ad}}) \cong \Sigma^\infty \mathcal{G}(\mathcal{P})$$

and the nth layer is the cofiber of

$$C(M; n)^{-TC(M;n)} \longrightarrow \mathcal{C}(\mathcal{P}^{\mathrm{Ad}}; n)^{-TC(M;n)}$$

This relates the stable homotopy type of the gauge group $\mathcal{G}(\mathcal{P})$ to Thom spectra of configuration spaces.

By the Thom isomorphism, the homology of $\mathcal{C}(\mathcal{P}^{\mathrm{Ad}}; n)^{-TC(M;n)}$ is the same as the homology of the base space $\mathcal{C}(\mathcal{P}^{\mathrm{Ad}}; n)$, with coefficients twisted by the orientation bundle of $C(M; n)$ pulled back to $\mathcal{C}(\mathcal{P}^{\mathrm{Ad}}; n)$. We can calculate this homology using the zig-zag of homotopy pullback squares

$$\begin{tikzcd}
G^n \arrow[d] & G^n \arrow[d] & G^n \arrow[d] \\
\mathcal{C}(\mathcal{P}^{\mathrm{Ad}}; n) \arrow[d] & \mathcal{F}(\mathcal{P}^{\mathrm{Ad}}; n) \arrow[l, "/\Sigma_n"'] \arrow[r] \arrow[d] \arrow[dl, phantom, "\llcorner"] \arrow[dr, phantom, "\lrcorner"] & (\mathcal{P}^{\mathrm{Ad}})^n \arrow[d] \\
C(M; n) & F(M; n) \arrow[l, "/\Sigma_n"'] \arrow[r] & M^n
\end{tikzcd}$$

where $F(M; n) \cong M^n - \Delta$ is ordered configurations of n points in M. Note that the manifold $F(M; n)$ is orientable iff M is orientable, while $C(M; n)$ is orientable iff M is orientable and $\dim M$ is even.

Another approach to understanding the homology of configuration spaces comes from the scanning map

$$C(M;n) \longrightarrow \Gamma_M(S^{TM})_n$$
$$\mathcal{C}(\mathcal{P}^{\mathrm{Ad}};n) \longrightarrow \Gamma_M(S^{TM} \wedge_M (\mathcal{P}^{\mathrm{Ad}} \amalg M))_n$$

Here the subscript of n denotes sections that are degree n in the appropriate sense. If M is open, the scanning map gives an isomorphism on integral homology in a stable range [31]. If M is closed, it gives an isomorphism on rational homology in a stable range [9].

4.2. **String Topology.** Now we will finally construct the tower we described in the introduction. We may start with the tower of section 3 and set $B = M$, $E = LM \amalg M$, and $X = M \amalg M$. Or, we may take the tower from section 4.1 and set $G \simeq \Omega M$ and $\mathcal{P} \simeq *$, so that $\mathcal{P}^{\mathrm{Ad}} \simeq LM$. Either construction gives the tower

$$\begin{array}{rcccc}
F(M \amalg M) & = & \Sigma^\infty \Gamma_M(LM \amalg M) & \simeq & \Sigma^\infty_+ \Omega_{\mathrm{id}} \mathrm{haut}(M) \\
 & & \vdots & & \\
P_n F(M \amalg M) & = & \Gamma_{(\mathbf{M}_n^{\mathrm{op}}, \{M^i\})}(\Sigma^\infty_{M^i} LM^i \amalg M^i) & & \\
 & & \vdots & & \\
P_2 F(M \amalg M) & = & \Gamma_{M \times M}(\Sigma^\infty_{M \times M} LM^2 \amalg M^2)^{\Sigma_2} & & \\
P_1 F(M \amalg M) & = & \Gamma_M(\Sigma^\infty_M LM \amalg M) & \simeq & LM^{-TM} \\
P_0 F(M \amalg M) & = & * & &
\end{array}$$

The nth layer is

$$D_n F(M \amalg M) \simeq \mathcal{C}(LM;n)^{-TC(M;n)}$$

As before, $\mathcal{C}(LM;n)$ is configurations of n unmarked free loops in M with distinct basepoints. This proves Theorem 1.2 from the introduction.

Note that the connectivity of $\mathcal{C}(LM;n)^{-TC(M;n)}$ decreases as $n \longrightarrow \infty$, so the tower does not converge. The homology of the first layer LM^{-TM} can be calculated using methods from [13]. Taking $M = S^n$, we can use known methods to calculate the second layer in rational homology

$$H_*(\mathcal{C}(LS^n;2)^{-TC(S^n;2)};\mathbb{Q})$$

Here is one recipe that works:

- Take the differentials for the Serre spectral sequence of

$$\Omega S^n \times \Omega S^n \longrightarrow LS^n \times LS^n \longrightarrow S^n \times S^n$$

  and pull them back to get the differentials for

$$\Omega S^n \times \Omega S^n \longrightarrow \mathcal{F}(LS^n;2) \longrightarrow F(S^n;2) \simeq S^n$$

  and get the rational homology of $\mathcal{F}(LS^n;2)$ with the Σ_2-action.

- Calculate the twisted rational homology of $\mathbb{RP}^\infty$ using
$$\mathbb{Z}/2 \longrightarrow S^\infty \longrightarrow \mathbb{RP}^\infty$$
- Calculate the rational homology of $\mathcal{C}(LS^n;2)$ using
$$\mathcal{F}(LS^n;2) \longrightarrow \mathcal{C}(LS^n;2) \longrightarrow \mathbb{RP}^\infty$$
- If n is odd, the manifold $C(S^n;2)$ is not orientable, so we need rational homology with twisted coefficients. Get this by subtracting off the rational homology of $\mathcal{C}(LS^n;2)$ from the rational homology of $\mathcal{F}(LS^n;2)$. This method is justified by examining the spectral sequence for
$$\mathbb{Z}/2 \longrightarrow \mathcal{F}(LS^n;2) \longrightarrow \mathcal{C}(LS^n;2)$$

If n is odd, then $H_q(\mathcal{C}(LS^n;2);\mathbb{Q})$ with twisted coefficients is
$$\begin{cases} \mathbb{Q} & q = n-1, 2n-2, 2n-1, 3n-2 \\ \mathbb{Q}^2 & q = 3n-3, 4n-4, 4n-3, 5n-4 \\ \vdots & \vdots \\ 0 & \text{otherwise} \end{cases}$$
and if n is even the answer (with untwisted coefficients) is
$$\begin{cases} \mathbb{Q} & q = n-1, 3n-3, 3n-2, 4n-4, \\ & \quad 5n-4, 6n-6, 6n-5, 8n-7 \\ \mathbb{Q}^2 & q = 5n-5, 7n-7, 7n-6, 8n-8, 9n-9, \\ & \quad 9n-8, 10n-10, 10n-9, 12n-11 \\ \vdots & \vdots \\ 0 & \text{otherwise} \end{cases}$$
To get the homology of $\mathcal{C}(LS^n;2)^{-TC(S^n;2)}$ we subtract $2n$ from each degree. This Thom spectrum is a homotopy fiber of maps of rings, so its homology carries an associative multiplication but no unit. When n is odd, all the products are zero. When n is even, many products are zero but certainly not all of them.

5. First Construction of P_nF

We still need to add teeth to Cor. 2.6 above by furnishing a functorial construction of P_nF for general F with the desired properties. We begin with a description of P_nF in the non-fiberwise case ($B = *$) that the author learned from Greg Arone. Broadly, this is the cellular approach and our second construction below is the simplicial approach.

Let $F : \mathcal{T}_{\text{fin}}^{\text{op}} \longrightarrow \mathcal{T}$ be a contravariant homotopy functor from finite based spaces to based spaces. We want to construct another functor P_nF that agrees with F on the spaces $\mathbf{R}_{*,n}$ with at most n points. A reasonable guess is to take a Kan extension

from $\mathbf{R}_{*,n}$ back to all of $\mathcal{T}_{\text{fin}}$. In fact, if we assume in addition that F is *topological* (enriched over spaces) and that we take the homotopy right Kan extension over topological functors, then we get the right answer.

We can give P_nF more explicitly as follows. Let $\mathbf{G}_n = \mathbf{R}_{*,n}$ be the category of finite based sets $\underline{i}_+ = \{1, \ldots, i\}_+$ with $0 \leq i \leq n$ and based maps between them. For a fixed finite based space X, define two diagrams of unbased spaces over $\mathbf{G}_n^{\text{op}}$:

$$\underline{i}_+ \rightsquigarrow X^i = \text{Map}_*(\underline{i}_+, X)$$

$$\underline{i}_+ \rightsquigarrow F(\underline{i}_+)$$

Then consider the space of (unbased!) maps between these two diagrams

$$P_nF(X) \stackrel{?}{=} \text{Map}_{\mathbf{G}_n^{\text{op}}}(X^i, F(\underline{i}_+))$$

Note that since F is topological, there is a natural map from $F(X)$ into this space. Furthermore, this map is a homeomorphism when $X = \underline{i}_+$ for $0 \leq i \leq n$, since then the diagram X^i is generated freely by a single point at level $\underline{i}_+$, corresponding to the identity map of $\underline{i}_+$. This is good, but we have missed the mark a little bit because this construction is not n-excisive in general.

To fix this, we take the *derived* or *homotopically correct* mapping space of diagrams instead. We could do this by fixing a model structure on $\mathbf{G}_n^{\text{op}}$ diagrams in which the weak equivalences are defined objectwise. Then we would replace $\{X^i\}$ by a cofibrant diagram and $\{F(\underline{i}_+)\}$ by a fibrant diagram. The space of maps between these replacements is by definition the homotopically correct mapping space.

More concretely, we can fatten up the diagram $\{X^i\}$ to the diagram

$$\underline{i}_+ \rightsquigarrow \underset{\underline{j}_+ \in (\mathbf{G}_n^{\text{op}}, \underline{i}_+)}{\text{hocolim}} X^j$$

and leave $\{F(\underline{i}_+)\}$ alone. Then the above conditions are satisfied for the projective model structure defined above in section 3.1. This standard thickening is sometimes called a *two-sided bar construction* [28] [35].

Equivalently, we can leave $\{X^i\}$ alone and fatten up $\{F(\underline{i}_+)\}$ to

$$\underline{i}_+ \rightsquigarrow \underset{\underline{j}_+ \in (\underline{i}_+, \mathbf{G}_n^{\text{op}})}{\text{holim}} F(\underline{j}_+)$$

Then the above conditions would be satisfied for the injective model structure, if it existed. Note that the two spaces we get in either case are actually homeomorphic:

$$P_nF(X) = \text{Map}_{\mathbf{G}_n^{\text{op}}}\left[\underset{\underline{j}_+ \in (\mathbf{G}_n^{\text{op}}, \underline{i}_+)}{\text{hocolim}} X^j, F(\underline{i}_+)\right] \cong \text{Map}_{\mathbf{G}_n^{\text{op}}}\left[X^j, \underset{\underline{i}_+ \in (\underline{j}_+, \mathbf{G}_n^{\text{op}})}{\text{holim}} F(\underline{i}_+)\right]$$

Take either of these as our definition of $P_nF(X)$. The natural map $F(X) \longrightarrow P_nF(X)$ can be seen by taking the previous case and observing in addition that there are always natural maps

$$\text{hocolim} \longrightarrow \text{colim} \qquad \text{or} \qquad \lim \longrightarrow \text{holim}$$

Consider the second description

$$P_n F(X) \cong \mathrm{Map}_{\mathbf{G}_n^{\mathrm{op}}}\left[X^j, \operatorname*{holim}_{\underline{i}_+ \in (\underline{j}_+, \mathbf{G}_n^{\mathrm{op}})} F(\underline{i}_+)\right]$$

Now it is clear that

$$P_n F(\underline{i}_+) \xrightarrow{\sim} F(\underline{i}_+) \xrightarrow{\sim} \operatorname*{holim}_{\underline{j}_+ \in (\underline{i}_+, \mathbf{G}_n^{\mathrm{op}})} F(\underline{j}_+)$$

under $F(\underline{i}_+)$, so $F \longrightarrow P_n F$ is an equivalence on spaces with at most n points.

Next, consider the first description

$$P_n F(X) = \mathrm{Map}_{\mathbf{G}_n^{\mathrm{op}}}\left[\operatorname*{hocolim}_{\underline{j}_+ \in (\mathbf{G}_n^{\mathrm{op}}, \underline{i}_+)} X^j, F(\underline{i}_+)\right]$$

Remembering that X is a CW complex, the diagram on the left is a CW complex of diagrams. It has one free cell of dimension $d+m$ at the vertex $\underline{i}_+$ for every choice of d-cell in X^j and choice of m-tuple of composable arrows

$$\underline{j}_+ = \underline{i_0}_+ \longrightarrow \underline{i_1}_+ \longrightarrow \ldots \longrightarrow \underline{i_m}_+ = \underline{i}_+$$

Therefore the techniques of section 3.2 above tell us that $P_n F$ is n-excisive. This completes the proof that n-excisive approximations exist for topological functors from based spaces to based spaces.

The assumption that F is topological is not a very strong one, assuming that F is a homotopy functor. To see this, first define Δ_X^{nd} as below (6.5) as the category of nondegenerate simplices $\Delta^p \longrightarrow X$. A map from $\Delta^p \longrightarrow X$ to $\Delta^q \longrightarrow X$ is a factorization $\Delta^q \hookrightarrow \Delta^p \longrightarrow X$, where $\Delta^q \hookrightarrow \Delta^p$ is a composition of inclusions of faces. The classifying space $|\Delta_X^{\mathrm{nd}}|$ is homeomorphic to the thin geometric realization of X.

Even when F is not topological, each map $\Delta_+^p \wedge Y \longrightarrow X$ gives a map

$$\Delta_+^p \wedge F(X) \xrightarrow{\sim} F(X) \longrightarrow F(\Delta_+^p \wedge Y)$$

which assemble into a zig-zag

$$F(X) \longrightarrow \operatorname*{holim}_{\Delta^{\mathrm{nd}}_{\mathrm{Map}_*(Y,X)}} F(\Delta_+^p \wedge Y) \xleftarrow{\sim} \operatorname*{holim}_{\Delta^{\mathrm{nd}}_{\mathrm{Map}_*(Y,X)}} F(Y)$$

$$\cong \mathrm{Map}(|\Delta^{\mathrm{nd}}_{\mathrm{Map}_*(Y,X)}|, F(Y)) \xleftarrow{\sim} \mathrm{Map}(\mathrm{Map}_*(Y,X), F(Y))$$

assuming $\mathrm{Map}_*(Y,X)$ has the homotopy type of a CW complex. So we don't quite get a map from $F(X)$ to the far right-hand side, but we get something close enough for the purposes of homotopy theory. In particular, setting $Y = \underline{i}_+$ we get a natural zig-zag

$$F(X) \longrightarrow \ldots \longrightarrow \mathrm{Map}(X^i, F(\underline{i}_+))$$

and therefore we get a natural zig-zag from $F(X)$ to $P_n F(X)$, which gives a natural map $F \longrightarrow P_n F$ in the homotopy category of functors. This map is still an equivalence when X has at most n points, because in that case the homotopy limits become ordinary products and we can use the same argument as above.

So much for the assumption that F is topological. Once we have the case where F takes based spaces to spaces, we can also easily handle the case when F takes based spaces to spectra. Simply post-compose F with fibrant replacement of spectra, and work one level at a time. This works because every stable equivalence of fibrant spectra gives a weak equivalence of spaces on each level. The above constructions naturally commute with taking the based loop space Ω, so they pass to a construction on spectra.

If F is defined on unbased spaces then we make the same construction, except that we replace $\mathbf{G}_n$ with the category of finite unbased sets $\mathbf{F}_n$. Using Prop. 2.5 above, we have finished the proof of the following:

Theorem 5.1. *If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a homotopy functor, where $\mathcal{C} = \mathcal{U}_{\text{fin}}$ or $\mathcal{T}_{\text{fin}}$ and $\mathcal{D} = \mathcal{T}$ or $\mathcal{S}p$, then there is a universal n-excisive approximation P_nF, and $F \longrightarrow P_nF$ is an equivalence on spaces with at most n points.*

We will reprove and extend this result in Theorem 7.1 below.

5.1. **On Fiberwise Spaces.** More generally, if F takes retractive spaces over B to spaces, then we expect $P_nF(X)$ to be some kind of mapping space from a functor represented by X into F. This is more difficult now because the various retractive spaces $\underline{i}_B = \underline{i} \amalg B$ have a topology. For each value of i, the collection of all spaces of the form $\underline{i}_B$ can be identified with

$$\mathrm{Map}(\underline{i}, B) \cong B^i$$

If X is a retractive space over B then X^i is a retractive space over B^i. So it is natural to look at maps from X^i to a bundle over B^i whose fiber is $F(\underline{i}_B)$. When $F(X)$ was $Q\mathrm{Map}_B(X, E)$, this bundle was $Q_{B^i}(E^i)$. In general, one might try to assume something stronger than F being topological, something that ensures the existence of a bundle

$$\begin{tikzcd} F(\underline{i}_B) \arrow[r] & \underline{F}(i) \arrow[d] \\ & B^i \end{tikzcd}$$

and a continuous map

$$F(X) \longrightarrow \mathrm{Map}_{B^i}(X^i, \underline{F}(i))$$

It seems more reasonable to assume only that F is a homotopy functor and to define a zig-zag of maps connecting the two together.

In addition, one needs to relate these mapping spaces together for varying i. The bundles are already handling the information of the map $\underline{i} \longrightarrow B$, so we throw that information out. Let $\mathbf{U}_{B,n}$ be the category of spaces *under* B of the form $\underline{i}_B = \underline{i} \amalg B$. That is, a map $\underline{i}_B \longrightarrow \underline{j}_B$ under B is required to send B to B by the identity map, but the points of $\underline{i}$ can go to any of the points in $\underline{j}$ or anywhere in B.

Note that $\mathbf{U}_{B,n}$ has a discrete set of objects, but the morphisms are enriched over spaces.

The spaces $\{X^i\}$ form an enriched diagram indexed by $\mathbf{U}_{B,n}$ that contains the diagram $\{B^i\}$ as a retract. So we should define our bundles $\underline{F}(i)$ so that they form a $\mathbf{U}_{B,n}$ diagram that also contains $\{B^i\}$ as a retract. Then we can define

$$P_n(X) \stackrel{?}{=} \mathrm{Map}_{(\mathbf{U}^{\mathrm{op}}_{B,n},\{B^i\})}(X^i, \underline{F}(i))$$

as the space of maps of topological diagrams over (but not under!) the diagram $\{B^i\}$. The question mark reminds us that, as before, we need to fatten up one side or the other to get the correct definition.

Unfortunately, we do not have a construction of the bundle $\underline{F}(i)$ that satisfies all of these properties. We will give a construction that falls a bit short of the mark, and use it to motivate a more simplicial construction of P_nF in 7.

First we construct a space over the simplicial fattening $|\Delta^{\mathrm{nd}}_{B^i}|$ of B^i, simply by taking the homotopy colimit of the diagram

$$\Delta^p \longrightarrow B^i \qquad \rightsquigarrow \qquad F((\Delta^p \times \underline{i}) \amalg B)$$

This space is a quasifibration ([33], p.98 and [16], 3.12) over the space

$$|\Delta^{\mathrm{nd}}_{B^i}| \cong \operatorname*{hocolim}_{\Delta^{\mathrm{nd}}_{B^i}} *$$

We want to push this forward along $f : |\Delta^{\mathrm{nd}}_{B^i}| \xrightarrow{\sim} B^i$ to a fibration over B^i. One way to do this is to replace the quasifibration by a fibration, then pull back along some homotopy inverse $g : B^i \longrightarrow |\Delta^{\mathrm{nd}}_{B^i}|$. This is good enough when dealing with just one bundle, but we would like a functorial construction to apply to all values of i.

A better approach is to observe that the pullback operation f^* has a left adoint $f_!$. Even more, the two form a Quillen equivalence between retractive spaces over $|\Delta^{\mathrm{nd}}_{B^i}|$ and retractive spaces over B^i. This is true for a few different model structures ([29], 7.3.5). Here we will take the q-model structure on $\mathcal{R}_B$, which is determined by the forgetful functor $\mathcal{R}_B \longrightarrow \mathcal{U}$. So a weak equivalence between fibrations over B induces an weak equivalence on each fiber. Therefore we take the q-cofibrant replacement of the bundle above, push it forward using $f_!$, and then take the q-fibrant replacement of the result. This gives a functorial construction of a fibration over B^i whose fiber has the homotopy type of $F(\underline{i}_B)$.

The drawback is that at least two of our constructions, the homotopy colimit over $\Delta^{\mathrm{nd}}_{B^i}$ and the q-cofibrant replacement, do not define continuous functors. Therefore we only get a natural map

$$F(X) \longrightarrow \mathrm{Map}_{(\mathbf{U}^{\mathrm{op}}_{B,n},\{B^i\})}(X^i, \underline{F}(i))$$

if we interpret the right-hand side as maps of unenriched diagrams, or equivalently if we interpret $\mathbf{U}_{B,n}$ as being discrete. This issue goes away if instead F is a

functor from unbased spaces over B, since we would replace $\mathbf{U}_{B,n}$ by the discrete category $\mathbf{F}_n$ of finite unbased sets with at most n points. Still, we will not finish this argument in that case since our second construction in section 7 below will take care of it.

To motivate this second approach, notice that we did not have to push forward from $|\Delta^{\mathrm{nd}}_{B^i}|$ to B^i. Instead, we could have looked at diagrams over $\{|\Delta^{\mathrm{nd}}_{B^i}|\}$ corresponding to X^i and $\underline{F}(i)$. Roughly, this is expressed as a homotopy limit

$$\operatorname*{holim}_{\Delta^p \times \underline{i} \longrightarrow X} F((\underline{i} \times \Delta^p)_B)$$

We will pursue this more in section 7.

5.2. **Higher Brown Representability.** Before we move on, we should point out that this first construction is better suited to proving a kind of Brown Representability for homogeneous n-excisive functors. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a homotopy functor as in Thm. 5.1 above; the source is based or unbased spaces and the target is based spaces or spectra. Then F is *n-reduced* if $P_{n-1}F \simeq *$, or equivalently if $F(X) \simeq *$ whenever X is a space with at most $(n-1)$ points. Note that

$$F_n(X) := \operatorname{hofib}(F(X) \longrightarrow P_{n-1}F(X))$$

is always n-reduced, and $F_n(\underline{n})$ is the *cross effect* $\operatorname{cross}_n F(\underline{1}, \ldots, \underline{1})$ defined below in section 8.

We say that F is *homogeneous n-excisive* if it is n-excisive and n-reduced. So $D_nF(X) = \operatorname{hofib}(P_nF(X) \longrightarrow P_{n-1}F(X))$ is always homogeneous n-excisive. Homogeneous 1-excisive functors are a good notion of space-valued or spectrum-valued cohomology theories. From numerous sources (e.g. [7], [14], [29]) we expect such cohomology theories to be represented by spaces or spectra.

Examining the construction of P_nF in this section, we see that $P_nF(X) \longrightarrow P_{n-1}F(X)$ is a Serre fibration when X is a CW complex. Therefore the ordinary fiber is equivalent to D_nF. This can be rephrased as the following:

Proposition 5.2.
- *If $F : \mathcal{U} \longrightarrow \mathcal{T}$ is an n-reduced homotopy functor then there is a natural map*

$$F(X) \longrightarrow D(X) := \operatorname{Map}_*(X^n/\Delta, F(\underline{n}))^{\Sigma_n}$$

in the homotopy category of functors on $\mathcal{U}$. If F is homogeneous n-excisive then this map is an equivalence.
- *If $F : \mathcal{T} \longrightarrow \mathcal{T}$ then the same is true for*

$$F(X) \longrightarrow D(X) := \operatorname{Map}_*(X^{\wedge n}/\Delta, F(\underline{n}_+))^{\Sigma_n}$$

- *Analogous statements hold when the target of F is spectra.*

Remark. If the source of F is $\mathcal{U}_{B,\mathrm{fin}}$ then the analogous statement should have

$$D(X) = \operatorname{Map}_{B^n}((X^n, \Delta), (\underline{F}(n), B^n))^{\Sigma_n}$$

and if the source is retractive spaces $\mathcal{R}_{B,\mathrm{fin}}$ then the standard form should be

$$D(X) = \mathrm{Map}_{B^n}((X^{\overline{\wedge} n}, B^n \cup \Delta), (\underline{F}(n), B^n))^{\Sigma_n}$$

where $\overline{\wedge}$ is the external smash product defined above in 3.2. We do not prove these statements here.

We will now strengthen this to an equivalence of homotopy categories. Let G be a finite group. Recall that the usual notion of *G-equivalence* of G-spaces is an equivariant map $X \longrightarrow Y$ which induces equivalences $X^H \longrightarrow Y^H$ for all subgroups $H < G$. We will call an equivariant map $X \longrightarrow Y$ a *naive G-equivalence* if it is merely an equivalence when we forget the G action. It is well known that there are at least two cofibrantly generated model structures on G-spaces, one which gives the G-equivalences and one which gives the naive G-equivalences.

Examining the behavior of $D(X)$ on the spaces $\underline{i}$ or $\underline{i}_+$ for $i \leq n$, it is clear that the homotopy type of $D(X)$ is determined by the *nonequivariant* or *naive* homotopy type of $F(\underline{n})$ or $F(\underline{n}_+)$. The following is then straightforward:

Proposition 5.3. *The above construction gives an equivalence between the homotopy category of homogeneous n-excisive functors to spaces and the naive homotopy category of Σ_n-spaces. A similar statement holds for functors to spectra.*

6. Properties of Homotopy Limits

In order to carry out our second construction of P_nF, we need a small collection of facts about homotopy limits. This section is expository except for Prop. 6.8.

Let $[n]$ denote the totally ordered set $\{0, 1, \ldots, n\}$ as a category. Let $\Delta[p]$ denote a p-simplex as a simplicial set, and let $\Delta^p = |\Delta[p]|$ denote its geometric realization. Recall [3] that if $A : \mathbf{I} \longrightarrow \mathbf{Top}$ is a diagram of spaces, the homotopy limit is defined to be

$$\operatorname*{holim}_{\mathbf{I}} A \subset \prod_{g:\Delta[n] \longrightarrow N\mathbf{I}} \mathrm{Map}_*(\Delta^n_+, A(g(n))),$$

the subset of all collections of maps that agree in the obvious way with the face and degeneracy maps of the nerve $N\mathbf{I}$. The following is perhaps the most standard result about homotopy limits, and we have already used it several times. It is included here for completeness.

Proposition 6.1. *If $A, B : \mathbf{I} \longrightarrow \mathcal{T}$ are two diagrams indexed by $\mathbf{I}$, and $A \longrightarrow B$ is a natural transformation that on each object $i \in \mathbf{I}$ gives a weak equivalence $A(i) \longrightarrow B(i)$, then it induces a weak equivalence*

$$\operatorname*{holim}_{\mathbf{I}} A \longrightarrow \operatorname*{holim}_{\mathbf{I}} B$$

Recall that if $\mathbf{I} \xrightarrow{\alpha} \mathbf{J}$ is a functor and $A : \mathbf{J} \longrightarrow \mathcal{T}$ is a diagram of spaces, then there is a naturally defined map

$$\operatorname*{holim}_{\mathbf{J}} A \longrightarrow \operatorname*{holim}_{\mathbf{I}} (A \circ \alpha)$$

The functor $\mathbf{I} \xrightarrow{\alpha} \mathbf{J}$ is *homotopy initial* (or *homotopy left cofinal*) if for each object $j \in \mathbf{J}$ the overcategory $(\alpha \downarrow j)$ has contractible nerve. Then we have the following pair of standard facts:

Proposition 6.2.
- *If $\mathbf{I} \xrightarrow{\alpha} \mathbf{J}$ is homotopy initial and $A : \mathbf{J} \longrightarrow \mathcal{T}$ is a diagram of spaces, then*

  $$\operatorname*{holim}_{\mathbf{J}} A \longrightarrow \operatorname*{holim}_{\mathbf{I}} (A \circ \alpha)$$

  is an equivalence.
- *If $\mathbf{I} \xrightarrow{\alpha} \mathbf{J}$ is the inclusion of a subcategory and $A : \mathbf{J} \longrightarrow \mathcal{T}$, then*

  $$\operatorname*{holim}_{\mathbf{J}} A \longrightarrow \operatorname*{holim}_{\mathbf{I}} (A \circ \alpha)$$

  is a Serre fibration.

In practice, we use these easy lemmas to check that a given functor α is homotopy initial:

Lemma 6.3. *Each adjunction of categories induces a homotopy equivalence on the nerves.*

Corollary 6.4.
- *If $(\alpha \downarrow j)$ is related by a zig-zag of adjunctions to the one-point category $*$, then its nerve is contractible and therefore α is homotopy initial.*
- *If $(\alpha \downarrow j)$ has an initial or terminal object then α is homotopy initial.*
- *If α is a left adjoint then it is homotopy initial.*

Here's an interesting example of a homotopy initial functor that we will use frequently:

Definition 6.5.
- If X is a space, let Δ_X^{nd} denote the category of nondegenerate simplices $\Delta^p \longrightarrow X$. A map from $\Delta^p \longrightarrow X$ to $\Delta^q \longrightarrow X$ is a factorization $\Delta^q \hookrightarrow \Delta^p \longrightarrow X$, where $\Delta^q \hookrightarrow \Delta^p$ is a composition of inclusions of faces. The classifying space of $\Delta_{X^i}^{\mathrm{nd}}$ is homeomorphic to the thin geometric realization of X^i:

  $$B\Delta_{X^i}^{\mathrm{nd}} \cong |\mathrm{sd}(S.(X^i))| \cong |S.(X^i)| \cong |(S.X)^i| \cong |S.X|^i$$
- Let Δ_X be the category of all (possibly degenerate) simplices in X, with face and degeneracy maps between them. Then the inclusion $\Delta_X^{\mathrm{nd}} \longrightarrow \Delta_X$ is a left adjoint, therefore homotopy initial.
- If $X.$ is a simplicial set, there are obvious analogues of $\Delta_{X.}^{\mathrm{nd}}$ and $\Delta_{X.}$. As before, the inclusion $\Delta_{X.}^{\mathrm{nd}} \hookrightarrow \Delta_{X.}$ is a left adjoint, therefore homotopy initial.

The next fact is about iterated homotopy limits; we recall the colimit version first. If $F : \mathbf{I} \longrightarrow \mathbf{Cat}$ is a small diagram of small categories, the *Grothendieck construction* gives a larger category $\mathbf{I} \int F$, whose objects are pairs (i, x) of an object $i \in \mathbf{I}$ and an object $x \in F(i)$. The maps $(i, x) \longrightarrow (j, y)$ are arrows $i \xrightarrow{f} j$ in $\mathbf{I}$, and arrows $F(f)(x) \longrightarrow y$ in $F(j)$. Then *Thomason's Theorem* is a standard fact that a homotopy colimit of a diagram $A : \mathbf{I} \int F \longrightarrow \mathcal{T}$ is expressed as an iterated homotopy colimit:

$$\operatorname*{hocolim}_{\mathbf{I} \int F} A \simeq \operatorname*{hocolim}_{i \in \mathbf{I}} \left(\operatorname*{hocolim}_{F(i)} A \right)$$

To formulate the result for homotopy limits, we again let $F : \mathbf{I} \longrightarrow \mathbf{Cat}$ be a small diagram of small categories. Then the *reverse Grothendieck construction* gives a larger category $\mathbf{I} \int^R F$, whose objects are again pairs (i, x) of an object $i \in \mathbf{I}$ and an object $x \in F(i)$. The maps $(i, x) \longrightarrow (j, y)$ are arrows $j \xrightarrow{f} i$ in $\mathbf{I}$, and arrows $x \longrightarrow F(f)(y)$ in $F(i)$. Note that this is related to the original Grothendieck construction in that

$$\mathbf{I} \int^R F \cong (\mathbf{I} \int (\mathrm{op} \circ F))^{\mathrm{op}}$$

Proposition 6.6 (Dual of Thomason's Theorem)**.** *For a diagram $A : \mathbf{I} \int^R F \longrightarrow \mathcal{T}$ then there is a natural weak equivalence*

$$\operatorname*{holim}_{\mathbf{I} \int^R F} A \xrightarrow{\sim} \operatorname*{holim}_{i \in \mathbf{I}^{\mathrm{op}}} \left(\operatorname*{holim}_{F(i)} A \right)$$

This is known and a careful proof is given in [34]. The proof starts with the observation that the big homotopy limit is a homotopy right Kan extension, which is a composite of two homotopy right Kan extensions:

$$\operatorname*{holim}_{\mathbf{I} \int^R F} A \simeq \operatorname*{holim}_{i \in \mathbf{I}^{\mathrm{op}}} \left(\operatorname*{holim}_{(j^{\mathrm{op}} \downarrow \Pi)} A \right)$$

where $\Pi : \mathbf{I} \int^R F \longrightarrow \mathbf{I}^{\mathrm{op}}$ is the obvious forgetful functor. Then we can show that the functor

$$F(j) \longrightarrow (j^{\mathrm{op}} \downarrow \Pi)$$

is a left adjoint, therefore homotopy initial, so the homotopy limit on the inside becomes $\operatorname*{holim}_{F(i)} A$. Strictly speaking, this only shows that there is a zig-zag of weak equivalences connecting $\operatorname*{holim}_{\mathbf{I} \int^R F} A$ to $\operatorname*{holim}_{i \in \mathbf{I}^{\mathrm{op}}} \left(\operatorname*{holim}_{F(i)} A \right)$. The obvious map between them does not make this zig-zag strictly commute, though Schlictkrull shows in [34] that it commutes up to homotopy. That's good enough to show that the natural map is also a weak equivalence.

In practice, we will come upon homotopy limits that are indexed by forwards Grothendieck constructions $\mathbf{I} \int F$ instead of reverse ones. Here we can still prove that the homotopy limit splits, but the result is more complicated.

Definition 6.7. If $\mathbf{I}$ is a category, the *twisted arrow category* $a\mathbf{I}$ has as its objects the arrows $i \longrightarrow j$ of $\mathbf{I}$. The morphisms from $i \longrightarrow j$ to $k \longrightarrow \ell$ are the factorizations of $k \longrightarrow \ell$ through $i \longrightarrow j$:

$$\begin{tikzcd} i \arrow[d] & k \arrow[l] \arrow[d] \\ j \arrow[r] & \ell \end{tikzcd}$$

Proposition 6.8. *Given a diagram $A : \mathbf{I} \int F \longrightarrow \mathcal{T}$ there is a natural weak equivalence*

$$\operatorname*{holim}_{\mathbf{I} \int F} A \xrightarrow{\sim} \operatorname*{holim}_{(i \xrightarrow{f} j) \in a\mathbf{I}} \left(\operatorname*{holim}_{F(i)} A \circ F(f) \right)$$

Remark. This result is motivated by a result of Dwyer and Kan on function complexes [17]. Roughly, the left-hand side is the space of maps between two diagrams indexed by $\mathbf{I}$. The first diagram sends i to the nerve of $F(i)$, while the other sends i to $A(i)$. Mapping spaces of this form, if they are "homotopically correct," are equivalent to a homotopy limit of mapping spaces $\mathrm{Map}(NF(i), A(j))$ over the twisted arrow category $a\mathbf{I}$; this is roughly what we get on the right-hand side.

Proof. Recall that we already have a functor $F : \mathbf{I} \longrightarrow \mathbf{Cat}$. Define another functor $(a\mathbf{I})^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ by taking $i \longrightarrow j$ to $F(i)$, and call this functor F by abuse of notation. Then we can build the reverse Grothendieck construction $(a\mathbf{I})^{\mathrm{op}} \int^R F$.

The desired weak equivalence is the composite

$$\operatorname*{holim}_{\mathbf{I} \int F} A \xrightarrow{\sim} \operatorname*{holim}_{(a\mathbf{I})^{\mathrm{op}} \int^R F} A \circ \alpha \xrightarrow{\sim} \operatorname*{holim}_{(i \xrightarrow{f} j) \in a\mathbf{I}} \left(\operatorname*{holim}_{F(i)} A \circ F(f) \right)$$

The second map is a weak equivalence by the dual of Thomason's theorem, stated above. The first map is induced by pullback along a functor

$$(a\mathbf{I})^{\mathrm{op}} \int^R F \xrightarrow{\alpha} \mathbf{I} \int F$$

and it suffices to show that this functor is homotopy initial. Specifically, α does the following to objects and morphisms:

$$\begin{tikzcd}
(i \arrow[r, "f"] & j, \arrow[dd, "h"] & x \in F(i)) \arrow[d, "\varphi"] \arrow[r, squiggly, "\alpha"] & (j, \arrow[dd, "h"] & F(f)(x) \in F(j)) \arrow[d, squiggly, "F(h)"] \\
& & F(g)(x') \in F(i) & & F(hf)(x) \in F(j') \arrow[d, "F(hf)\varphi"] \\
(i' \arrow[uu, "g"] \arrow[r, "f'"] & j', & x' \in F(i')) \arrow[u, squiggly, "F(g)"] \arrow[r, squiggly, "\alpha"] & (j', & F(f')(x') \in F(j'))
\end{tikzcd}$$

Fix an object $(\ell, z \in F(\ell))$ in the target category $I \int F$. We'll show that the overcategory $(\alpha \downarrow (\ell, z))$ is contractible. A typical map between objects of this overcategory is given by the data

$$\begin{tikzcd}
(i \arrow[r, "f"] & j \arrow[r, "p"] \arrow[dd, "h"] & \ell, \arrow[dd, equal] & x \in F(i) \arrow[r, squiggly, "F(pf)"] \arrow[d, "\varphi"] & F(pf)(x) \arrow[r, "\sigma"] \arrow[dd, "F(pf)\varphi"] & z) \arrow[dd, equal] \\
 & & & F(g)(x') \in F(i) \arrow[dr, squiggly, "F(pf)"] & & \\
(i' \arrow[r, "f'"] \arrow[uu, "g"] & j' \arrow[r, "p'"] & \ell, & x' \in F(i') \arrow[u, squiggly, "F(g)"] \arrow[r, squiggly, "F(p'f')"] & F(p'f')(x') \arrow[r, "\sigma'"] & z)
\end{tikzcd}$$

where everything commutes. Let $\mathbf{J}$ be the subcategory of $(\alpha \downarrow (\ell, z))$ consisting of objects for which $j = \ell$ and p is the identity. Then there is a projection $P : (\alpha \downarrow (\ell, z)) \longrightarrow \mathbf{J}$ which is left adjoint to the inclusion $I : \mathbf{J} \longrightarrow (\alpha \downarrow (\ell, z))$. We can exhibit P and the natural transformation from the identity to $I \circ P$ in the following diagram:

$$\begin{tikzcd}
(i \arrow[r, "f"] \arrow[dd, equal] & j \arrow[r, "p"] \arrow[dd, "p"] & \ell, \arrow[dd, equal] & x \in F(i) \arrow[r, squiggly, "F(pf)"] \arrow[d, equal] & F(pf)(x) \arrow[r, "\sigma"] \arrow[dd, equal] & z) \arrow[dd, equal] \\
 & & & x \in F(i) \arrow[dr, squiggly, "F(pf)"] & & \\
(i \arrow[r, "pf"] & \ell \arrow[r, equal] & \ell, & x \in F(i) \arrow[u, squiggly, "\mathrm{id}"] \arrow[r, squiggly, "F(pf)"] & F(pf)(x) \arrow[r, "\sigma"] & z)
\end{tikzcd}$$

To check the adjunction, it suffices to check that a map from any object of $(\alpha \downarrow (\ell, z))$ into an object of $\mathbf{J}$ factors uniquely through this projection. Once this is checked, the next step is to show that $\mathbf{J}$ has an initial subcategory $\mathbf{K}$. A typical object of $\mathbf{K}$ is given in the first row below.

$$\begin{tikzcd}
(\ell \arrow[r, equal] & \ell \arrow[r, equal] \arrow[dd, equal] & \ell, \arrow[dd, equal] & F(f)(x) \in F(\ell) \arrow[r, squiggly, "\mathrm{id}"] \arrow[d, equal] & F(f)(x) \arrow[r, "\sigma"] \arrow[dd, equal] & z) \arrow[dd, equal] \\
 & & & F(f)(x) \in F(\ell) \arrow[dr, squiggly, "\mathrm{id}"] & & \\
(i \arrow[r, "f"] \arrow[uu, "f"] & \ell \arrow[r, equal] & \ell, & x \in F(i) \arrow[u, squiggly, "F(f)"] \arrow[r, squiggly, "F(f)"] & F(f)(x) \arrow[r, "\sigma"] & z)
\end{tikzcd}$$

The rest of the diagram justifies the claim that $\mathbf{K}$ is initial. Finally, $\mathbf{K}$ is isomorphic to the category of objects over z in $F(\ell)$, which has terminal object z. We have completed a zig-zag of adjunctions between $(\alpha \downarrow (\ell, z))$ and $*$, so $(\alpha \downarrow (\ell, z))$ is contractible. Therefore α is homotopy initial and the equivalence is complete.

The equivalence is clearly natural in A, but it is also natural in F in the following sense. A map of diagrams of categories $F \xrightarrow{\eta} G$ gives a map $\mathbf{I} \int F \xrightarrow{\mathbf{I} \int \eta} \mathbf{I} \int G$, so

a diagram $A : \mathbf{I} \int G \longrightarrow \mathcal{T}$ can be pulled back to $\mathbf{I} \int F$. Our equivalence then fits into a commuting square:

$$\begin{tikzcd}
\operatorname*{holim}_{\mathbf{I} \int F} (\mathbf{I} \int \eta)^* A \arrow[r] & \operatorname*{holim}_{(i \xrightarrow{f} j) \in a\mathbf{I}} \left(\operatorname*{holim}_{F(i)} ((\mathbf{I} \int \eta)^* A) \circ F(f) \right) \\
\operatorname*{holim}_{\mathbf{I} \int G} A \arrow[r] \arrow[u] & \operatorname*{holim}_{(i \xrightarrow{f} j) \in a\mathbf{G}} \left(\operatorname*{holim}_{G(i)} A \circ G(f) \right) \arrow[u]
\end{tikzcd}$$

□

Lastly, we want a result on diagrams $A : \mathbf{J} \longrightarrow \mathcal{T}$ for which every arrow $i \longrightarrow j$ induces a weak equivalence $A(i) \longrightarrow A(j)$. Call such a diagram *almost constant.* Of course, if A is a constant diagram sending everything to the space X, then its homotopy limit is

$$\operatorname*{holim}_{\mathbf{J}} A = \operatorname{Map}(B\mathbf{J}, X)$$

where $B\mathbf{J} = |N\mathbf{J}|$ is the classifying space of $\mathbf{J}$. If A is instead almost constant, then we get (see [14], [16])

Proposition 6.9. *If $A : \mathbf{J} \longrightarrow \mathcal{T}$ is almost constant, then there is a fibration $E_A \longrightarrow B\mathbf{J}$ and a natural weak equivalence*

$$\operatorname*{holim}_{\mathbf{J}} A \simeq \Gamma_{B\mathbf{J}}(E_A)$$

Moreover, if $\mathbf{I} \xrightarrow{\alpha} \mathbf{J}$ is a functor then there is a homotopy pullback square

$$\begin{tikzcd}
E_{A\circ\alpha} \arrow[r] \arrow[d] & E_A \arrow[d] \\
B\mathbf{I} \arrow[r] & B\mathbf{J}
\end{tikzcd}$$

Corollary 6.10. *If $A : \mathbf{J} \longrightarrow \mathcal{T}$ is almost constant, and $\mathbf{I} \xrightarrow{\alpha} \mathbf{J}$ induces a weak equivalence $B\mathbf{I} \longrightarrow B\mathbf{J}$, then the natural map*

$$\operatorname*{holim}_{\mathbf{J}} A \longrightarrow \operatorname*{holim}_{\mathbf{I}} (A \circ \alpha)$$

is a weak equivalence.

7. Second Construction of P_nF: The Higher Coassembly Map

Here we will describe how to construct $P_nF(X)$ as a homotopy limit

$$P_nF(X) = \operatorname*{holim}_{\Delta^p \times \underline{i} \longrightarrow X} F(\underline{i} \times \Delta^p)$$

When $n = 1$ and F is reduced, this construction is essentially the same as the *coassembly map* described in [14]. The coassembly map is formally dual to the *assembly map* ([37]) often found in treatments of algebraic K-theory.

We will prove that our construction of P_nF satisfies four properties:

(1) P_nF is a homotopy functor.
(2) P_nF takes pushout cubes whose dimension is at least $n+1$ to Cartesian cubes.
(3) If X is a CW complex then $P_nF(X) \longrightarrow \underset{X' \subset X \text{ finite complex}}{\operatorname{holim}} P_nF(X')$ is an equivalence.
(4) $F \longrightarrow P_nF$ is an equivalence on $\mathbf{R}^{\mathrm{op}}_{B,n}$ or $\mathbf{O}^{\mathrm{op}}_{B,n}$.

For functors on finite CW complexes, conditions (1), (2), and (4) are enough to imply P_nF is the universal n-excisive approximation of F. Condition (3) is a bit weaker than the standard condition that filtered homotopy colimits go to homotopy limits; it is here because the technology we need for (2) happens to make (3) easy.

There are 8 different setups we might consider, based on whether B is a point or not, the spaces over B are fiberwise based (retractive) or unbased, and F goes into spaces or spectra. We will first handle all cases where the spaces over B are unbased. Then we'll handle all cases where $B = *$ and the spaces over B are based. Together this gives an extension and a second proof of Theorem 5.1 above:

Theorem 7.1. *If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a homotopy functor, where $\mathcal{C} = \mathcal{U}_{B,\mathrm{fin}}$ or $\mathcal{T}_{\mathrm{fin}}$ and $\mathcal{D} = \mathcal{T}$ or $\mathcal{S}p$, then there is a universal n-excisive approximation P_nF, and $F \longrightarrow P_nF$ is an equivalence on spaces with at most n points.*

Finally, in section 8.2 below we will do the case of functors from retractive spaces over B to spectra. We do not have a method that works for retractive spaces over B to spaces.

Here is the the general approach. If F is a functor into spaces, we define P_nF as a homotopy limit of values of F on spaces equivalent to those that have at most n points. It is approximately the same thing as restricting F to spaces with at most n points and then taking a right Kan extension back to all spaces; this is why we expect (4) to be true. Since homotopy limits commute, the construction commutes with Ω and so it yields a valid construction when F is a functor into spectra.

From the definition it is not obvious that P_nF is a homotopy functor. We use Prop. 6.8 to put P_nF into a form which is homotopy invariant by Prop. 6.10. Finally, (3) is an easy consequence of the dual of Thomason's theorem (Prop. 6.6), while (2) becomes an easy consequence once we transfer our constructions from spaces to simplicial sets.

7.1. P_nF **for Unbased Spaces over** B**.** Let $\mathbf{C}_{B,n}$ denote a subcategory of simplicial sets over $S.B$ consisting of objects of the form

$$\underline{i} \times \Delta[p], \qquad p \geq 0 \text{ and } 0 \leq i \leq n.$$

Specifically, we take one such object for each choice of p and i, *and* each choice of map of simplicial sets $\underline{i} \times \Delta[p] \longrightarrow S.B$. We do *not* take the full subcategory on these objects. Each map

$$\underline{j} \times \Delta[q] \longrightarrow \underline{i} \times \Delta[p]$$

must be a product of a single simplicial map $\Delta[q] \longrightarrow \Delta[p]$ and a map of finite sets $\underline{j} \longrightarrow \underline{i}$. Intuitively, $\mathbf{C}_{B,n}$ is a simplicial fattening of $\mathbf{O}_{B,n}$.

Now let F be any contravariant homotopy functor from unbased spaces over B to spaces or spectra. If F is a functor to spectra, compose it with fibrant replacement. This gives an equivalent functor that takes weak equivalences of spaces to level equivalences of spectra, and we can argue one level at a time. So now without loss of generality, F is a homotopy functor to based spaces.

If $X.$ is a simplicial set over $S.B$, define

$$P_n F(X.) = \operatorname*{holim}_{(\mathbf{C}_{B,n} \downarrow X.)^{\mathrm{op}}} F(\underline{i} \times \Delta^p)$$

Abusing notation, define $P_n F$ on spaces as the composite

$$\mathcal{U}_B \xrightarrow{S.} \mathbf{sSet}/S.B \xrightarrow{P_n F} \mathcal{T}$$

or more explicitly,

$$P_n F(X) = \operatorname*{holim}_{(\mathbf{C}_{B,n} \downarrow S.X)^{\mathrm{op}}} F(\underline{i} \times \Delta^p)$$

The natural transformation $F \xrightarrow{p_n} P_n F$ is then induced by a collection of maps $F(X) \longrightarrow F(\underline{i} \times \Delta^p)$ for each map $\underline{i} \times \Delta^p \longrightarrow X$.

When $X = \underline{i}$, the object $\underline{i} \times \Delta[0] \xrightarrow{\cong} S.\underline{i}$ is initial in $(\mathbf{C}_{B,n} \downarrow S.\underline{i})^{\mathrm{op}}$, so the homotopy limit is obtained by evaluating at this initial object (Prop. 6.2). This proves property (4), that $F \longrightarrow P_n F$ is an equivalence on $\mathbf{O}_{B,n}^{\mathrm{op}}$.

Next we'll tackle property (1), that $P_n F$ is a homotopy functor.

Definition 7.2. Let $\mathbf{F}_n = \mathbf{O}_{*,n}$ be the category of finite unbased sets $\underline{0}, \ldots, \underline{n}$ and all maps between them.

Notice that we can define a functor $\Delta : \mathbf{F}_n^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ taking $\underline{i}$ to $\Delta_{X^i_.}$. Each map $\underline{i} \longleftarrow \underline{j}$ goes to the functor $\Delta_{X^i_.} \longrightarrow \Delta_{X^j_.}$ arising from the map $X^i_. \longrightarrow X^j_.$, whose definition is obvious once we observe that $X^i \cong \mathrm{Map}(\underline{i}, X)$. Now take the forwards Grothendieck construction $\mathbf{F}_n^{\mathrm{op}} \int \Delta$. This is a category whose objects are elements X^i_p and whose morphisms $X^i_p \longrightarrow X^j_q$ are compositions of maps $X^i_. \longrightarrow X^j_.$ from above and maps $X^j_p \longrightarrow X^j_q$ which are compositions of face and degeneracy maps. Equivalently, the objects can be described as maps

$$\Delta[p] \times \underline{i} \longrightarrow X.$$

and the morphisms are factorizations

$$\begin{tikzcd} \Delta[p] \times \underline{i} \arrow[r] & X. \arrow[d, equal] \\ \Delta[q] \times \underline{j} \arrow[r] \arrow[u] & X. \end{tikzcd}$$

in which the vertical map is a product of $\underline{j} \longrightarrow \underline{i}$ and some simplicial map $\Delta[q] \longrightarrow \Delta[p]$. This is clearly the same category as $(\mathbf{C}_{B,n}^{\mathrm{op}} \downarrow X.)^{\mathrm{op}}$, so we have a new way to write our definition of $P_n F(X.)$:

$$P_n F(X.) = \operatorname*{holim}_{\mathbf{F}_n^{\mathrm{op}} \int \Delta} F(\underline{i} \times \Delta^p)$$

Now Prop. 6.8 gives the following:

$$\operatorname*{holim}_{\mathbf{F}_n \int \Delta_{X^i}} F(\Delta^p \times \underline{i}) \simeq \operatorname*{holim}_{(\underline{i} \leftarrow \underline{j}) \in a\mathbf{F}_n^{\mathrm{op}}} \left(\operatorname*{holim}_{\Delta_{X^i_\cdot}} F(\underline{j} \times \Delta^p) \right)$$

The term inside the parentheses can be rewritten

$$\operatorname*{holim}_{\Delta_{X^i_\cdot}} F(\underline{j} \times \Delta^p) \simeq \operatorname*{holim}_{\Delta^{\mathrm{nd}}_{X^i_\cdot}} F(\underline{j} \times \Delta^p)$$

and this defines a homotopy functor in $X.$ by Prop. 6.10. The homotopy limit of these is also a homotopy functor, and using the naturality statement in Prop. 6.8 we conclude that $P_n F(X.)$ is a homotopy functor. In fact, we have proven something stronger than (1), that $P_n F$ actually takes weak equivalences of simplicial sets to weak equivalences.

Now we can prove (2). From [20], each strongly co-Cartesian cube of spaces over B is weakly equivalent to a pushout cube formed by a cofibrant space A and an $(n+1)$-tuple of spaces $X_0, \ldots, X_n$ over B, each with a cofibration $A \longrightarrow X_i$. Applying singular simplices $S.$, we get a cube of simplicial sets

$$T \rightsquigarrow S. \left(\bigcup_{s \in T} X_s \right)$$

where the $\bigcup$ is shorthand for pushout of spaces along A. We can show that this cube is equivalent to the pushout cube of simplicial sets

$$T \rightsquigarrow \bigcup_{s \in T} S.X_s$$

where the $\bigcup$ is shorthand for pushout of simplicial sets along $S.A$. There is a natural map from the latter to the former. Inductively, this is an equivalence for a given subset $T \subset S$, and then we show that it is an equivalence for $T \cup \{t\} \subset S$ by

comparing these two homotopy pushout squares:

$$\begin{tikzcd} {|S.A|} \arrow[r] \arrow[d] & {|S.X_t|} \arrow[d] \\ \left|\bigcup_{s\in T} S.X_s\right| \arrow[r] & \left|\bigcup_{s\in T\cup\{t\}} S.X_s\right| \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & X_t \arrow[d] \\ \bigcup_{s\in T} X_s \arrow[r] & \bigcup_{s\in T\cup\{t\}} X_s \end{tikzcd}$$

The natural map from the first to the second is a weak equivalence on the first three vertices. Therefore the natural map is a weak equivalence on the last vertex as well. This equivalence factors as

$$\left|\bigcup_{s\in T\cup\{t\}} S.X_s\right| \longrightarrow \left|S.\bigcup_{s\in T\cup\{t\}} X_s\right| \xrightarrow{\sim} \bigcup_{s\in T\cup\{t\}} X_s$$

so by 2 out of 3, the desired map $\left|\bigcup_{s\in T\cup\{t\}} S.X_s\right| \longrightarrow \left|S.\bigcup_{s\in T\cup\{t\}} X_s\right|$ is an equivalence.

This completes the induction. We can replace our cube with a pushout cube of simplicial sets. Since P_nF is a homotopy functor on simplicial sets, applying P_nF to both cubes gives equivalent results. Therefore it suffices to show that P_nF takes a pushout cube of simplicial sets to a Cartesian cube of spaces.

So let S by any set with cardinality strictly larger than n, let $A \in \mathbf{sSet}$ be a simplicial set, and for each element $s \in S$, let $X_s \in \mathbf{sSet}$ be a simplicial set containing A. Then there is a pushout cube which assigns each subset $T \subset S$ to the simplicial set $\bigcup_{t\in T} X_t$, which is shorthand for the pushout of the X_t along A. We want to show that P_nF takes this to a Cartesian cube; in other words, the natural map

$$\operatorname*{holim}_{\Delta[p]\times\underline{i}\to\bigcup_S X_s} F(\underline{i}\times\Delta^p) \longrightarrow \operatorname*{holim}_{(T\subsetneq S)^{\mathrm{op}}} \left(\operatorname*{holim}_{\Delta[p]\times\underline{i}\to\bigcup_T X_s} F(\underline{i}\times\Delta^p)\right)$$

is an equivalence. Using dual Thomason, we rewrite the right-hand side as

$$\operatorname*{holim}_{(T,\Delta[p]\times\underline{i}\to\bigcup_T X_s)} F(\underline{i}\times\Delta^p)$$

where each object of the indexing category is a proper subset $T \subsetneq S$, integers $p \geq 0$ and $0 \leq i \leq n$, and a map $\Delta[p]\times\underline{i} \to \bigcup_T X_s$. A map between two objects looks like

$$\begin{tikzcd} T, & \underline{i}\times\Delta[p] \arrow[r] & \bigcup_T X_s \\ U, \arrow[u, "\text{subset}"] & \underline{j}\times\Delta[q] \arrow[r] \arrow[u] & \bigcup_U X_s \arrow[u] \end{tikzcd}$$

This category maps forward into $\mathbf{F}_n^{\mathrm{op}} \int \Delta_{(\bigcup_S X_s)^i}$, in which a map between two objects is given by the data

$$\begin{tikzcd}
\underline{i} \times \Delta[p] \arrow[r] & \bigcup_S X_s \\
\underline{j} \times \Delta[q] \arrow[r] \arrow[u] & \bigcup_S X_s \arrow[u]
\end{tikzcd}$$

This functor α forgets the data of T and includes $\bigcup_T X_s^i$ into $\bigcup_S X_s^i$. The natural map of homotopy limits

$$\underset{\Delta[p]\times\underline{i}\to\bigcup_S X_s}{\operatorname{holim}} F(\underline{i} \times \Delta^p) \longrightarrow \underset{(T,\Delta[p]\times\underline{i}\to\bigcup_T X_s)}{\operatorname{holim}} F(\underline{i} \times \Delta^p)$$

is induced by a pullback of diagrams along α, so we just have to show that α is homotopy initial. Given an object $\underline{j} \times \Delta[q] \xrightarrow{\varphi} \bigcup_S X_s$ in the target category, the overcategory $(\alpha \downarrow \varphi)$ has as its objects the factorizations of φ

$$\underline{j} \times \Delta[q] \longrightarrow \underline{i} \times \Delta[p] \longrightarrow \bigcup_T X_s \longrightarrow \bigcup_S X_s$$

where $T \subsetneq S$ must be a *proper* subset of S.

Let us give a terminal object for this overcategory. Since we are working with simplicial sets instead of spaces, each q-simplex lands *inside* one of the sets X_s in the pushout. Therefore there is a smallest subset $T \subset S$ such that $\underline{j} \times \Delta[q] \xrightarrow{\varphi} \bigcup_S X_s$ lands inside $\bigcup_T X_s$, and since $j \leq n < |S|$, this subset is proper. This gives a terminal object for the overcategory $(\alpha \downarrow \varphi)$, so it's contractible, which finishes (2).

Finally we check (3). Let X be a CW complex. We want to show that the natural map

$$\underset{\Delta[p]\times\underline{i}\to S.X}{\operatorname{holim}} F(\underline{i} \times \Delta^p) \longrightarrow \underset{(\text{finite}\, X'\subset X)^{\mathrm{op}}}{\operatorname{holim}} \left(\underset{\Delta[p]\times\underline{i}\to S.X'}{\operatorname{holim}} F(\underline{i} \times \Delta^p) \right)$$

is an equivalence. Using dual Thomason, we rewrite the right-hand side as

$$\underset{(\text{finite}\, X'\subset X, \Delta[p]\times\underline{i}\to S.X')}{\operatorname{holim}} F(\underline{i} \times \Delta^p)$$

where each object of the indexing category is a finite subcomplex $X' \subset X$, integers $p \geq 0$ and $0 \leq i \leq n$, and a map $\Delta[p] \times \underline{i} \to S.X'$. A map between two objects looks like

$$\begin{tikzcd}
X', & \underline{i} \times \Delta[p] \arrow[r] & S.X' \\
X'', \arrow[u, "\text{inclusion}"] & \underline{j} \times \Delta[q] \arrow[r] \arrow[u] & S.X'' \arrow[u]
\end{tikzcd}$$

This category maps forward into $\mathbf{F}_n^{\mathrm{op}} \int \Delta_{(S.X)^i}$, in which a map between two objects looks like

$$\begin{tikzcd} \underline{i} \times \Delta[p] \arrow[r] & S.X \\ \underline{j} \times \Delta[q] \arrow[r] \arrow[u] & S.X \arrow[u] \end{tikzcd}$$

This functor α forgets the data of X' and includes X' into X. The natural map of homotopy limits defined above is again induced by a pullback of diagrams along α, so we just have to show that α is homotopy initial. Given an object $\underline{j} \times \Delta[q] \xrightarrow{\varphi} S.X$ in the target category, the overcategory $(\alpha \downarrow \varphi)$ has as its objects the factorizations of φ

$$\underline{j} \times \Delta[q] \longrightarrow \underline{i} \times \Delta[p] \longrightarrow S.X' \longrightarrow S.X$$

where $X' \subset X$ must be a finite subcomplex. But of course each q-simplex lands inside a unique smallest subcomplex; taking the union over all $\underline{j}$ gives a smallest finite subcomplex containing the image of $\Delta^q \times \underline{j}$. This gives a terminal object for the overcategory $(\alpha \downarrow \varphi)$, so it's contractible and we are done proving (3).

7.2. $P_n F$ **for Based Spaces.** The argument mimics the one above, so we will only point out what is different. The category $\mathbf{C}_n$ becomes a subcategory of based simplicial sets consisting of objects of the form

$$(\underline{i} \times \Delta[p])_+, \qquad p \geq 0 \text{ and } 0 \leq i \leq n.$$

with one such object for each choice of p and i. Each map

$$(\underline{j} \times \Delta[q])_+ \longrightarrow (\underline{i} \times \Delta[p])_+$$

is a choice of simplicial map $\Delta[q] \longrightarrow \Delta[p]$ and map of finite based sets $\underline{j}_+ \longrightarrow \underline{i}_+$. Intuitively, $\mathbf{C}_n$ is a simplicial fattening of $\mathbf{R}_{*,n} \cong \mathbf{G}_n$. If $X.$ is a based simplicial set, define

$$P_n F(X.) = \operatorname*{holim}_{(\mathbf{C}_n \downarrow X.)^{\mathrm{op}}} F((\underline{i} \times \Delta^p)_+)$$

Abusing notation, define $P_n F$ on spaces as the composite

$$\mathcal{T} \xrightarrow{S.} \mathbf{sSet}_* \xrightarrow{P_n F} \mathcal{S}p$$

The category $\mathbf{F}_n$ of finite sets is replaced by the category $\mathbf{G}_n$ of finite based sets. As before, there is a functor $\Delta : \mathbf{U}_{*,n}^{\mathrm{op}} \cong \mathbf{G}_n^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ taking $\underline{i}_+$ to $\Delta_{X.^i}$, and we can rewrite $P_n F(X.)$ as

$$P_n F(X.) = \operatorname*{holim}_{\mathbf{G}_n^{\mathrm{op}} \int \Delta} F((\underline{i} \times \Delta^p)_+)$$

To show that $P_n F$ is homotopy invariant we rewrite it as

$$\operatorname*{holim}_{\mathbf{G}_n^{\mathrm{op}} \int \Delta} F(\Delta^p \times \underline{i}) \simeq \operatorname*{holim}_{(\underline{i}_+ \longleftarrow \underline{j}_+) \in a\mathbf{G}_n^{\mathrm{op}}} \left(\operatorname*{holim}_{\Delta_{X.^i}} F((\underline{j} \times \Delta^p)_+) \right)$$

which proves (1). The proof of (2) and (3) is the same as in the unbased case.

7.3. **Difficulties with Retractive Spaces over B.** The above proof fails for retractive spaces over B. The clearest generalization is to define $\mathbf{U}_{B,n}^{\mathrm{op}} \int \Delta$ and then to define $P_n F$ as a homotopy limit over this category. The proof of (1), (2) and (3) is then straightforward. However, (4) fails because there aren't enough maps in $\mathbf{U}_{B,n}^{\mathrm{op}} \int \Delta$ to make our desired object initial.

Examining the failure, it seems one must enrich $\mathbf{U}_{B,n}^{\mathrm{op}}$ and use an enriched version of the above theorems on homotopy limits. In order to define $P_n F$ here, one must deal with the concept of a "diagram" that is indexed not by a simplicially enriched category but by a simplicial object in **Cat**. We will avoid doing this, and instead we will handle the case of $F : \mathcal{R}_B \longrightarrow \mathcal{S}p$ in section 8.2 using splitting theorems that only hold for functors into spectra.

8. Spectra and Cross Effects

From here onwards we will only consider functors from retractive spaces over B to spectra. In this section the word *spectra* will refer to prespectra, though the arguments will also work for coordinate-free orthogonal spectra that have non-degenerately based levels [27]. Let fib denote homotopy fiber and cofib denote (reduced) homotopy cofiber. For spaces, these have the usual definition

$$\mathrm{fib}\,(X \longrightarrow Y) = X \times_Y \mathrm{Map}_*(I, Y)$$
$$\mathrm{cofib}\,(X \longrightarrow Y) = (X \wedge I) \cup_X Y$$

and for spectra these definitions are applied to each level separately.

We begin this section with some standard facts about spectra and splitting. Recall that the natural map $X \vee Y \longrightarrow X \times Y$ is an equivalence when X and Y are spectra. Comparison of cofiber and fiber sequences of spectra then gives the following:

Proposition 8.1. *Suppose that X, X', and Y are spectra with maps*

$$X \xrightarrow{i} Y \xrightarrow{p} X'$$

such that $p \circ i$ is an equivalence. Then there are natural equivalences of spectra

$$X \vee \mathrm{fib}\,(p) \xrightarrow{\sim} Y \xrightarrow{\sim} X \times \mathrm{cofib}\,(i)$$

which also yield an equivalence $\mathrm{fib}\,(p) \xrightarrow{\sim} \mathrm{cofib}\,(i)$.

Corollary 8.2. *If X is a retract of Y then $Y \simeq X \vee Z$ where*

$$Z \simeq \mathrm{fib}\,(Y \longrightarrow X) \simeq \mathrm{cofib}\,(X \longrightarrow Y)$$

Corollary 8.3. *If X is a well-based space then there is a natural equivalence*

$$\Sigma^\infty(X_+) \simeq \Sigma^\infty(X \vee S^0)$$

Corollary 8.4. *If $\mathcal{R}_B^{(\mathrm{op})} \xrightarrow{F} \mathcal{S}p$ is any covariant or contravariant functor then there is a splitting of functors*

$$F(X) \simeq F(B) \times \overline{F}(X)$$

where $\overline{F}(X)$ *can be defined as the fiber of* $F(X) \longrightarrow F(B)$ *or the cofiber of* $F(B) \longrightarrow F(X)$*. This also holds if* F *is only defined on finite CW complexes.*

We want a slight generalization of these results to n-dimensional cubes of retracts. First recall the higher-order versions of homotopy fiber and homotopy cofiber from [20]. If F is a n-cube of spectra then we can think of it as a map between two $(n-1)$-cubes. The *total homotopy fiber* $\mathrm{tfib}\,(F)$ is inductively defined as the homotopy fiber of the map between the total homotopy fibers of these two $(n-1)$-cubes. For a 0-cube consisting of the space X, we define the total fiber to be X. Therefore the total fiber of a 1-cube $X \longrightarrow Y$ is $\mathrm{fib}\,(X \longrightarrow Y)$.

The *total homotopy cofiber* $\mathrm{tcofib}\,(F)$ has a similar inductive definition. Recall from [20] that a cube is Cartesian iff its total fiber is weakly contractible, and co-Cartesian iff its total cofiber is weakly contractible. From this it quickly follows that a cube of spectra is Cartesian iff it is co-Cartesian.

If F is a functor $\mathcal{R}_B^{\mathrm{op}} \longrightarrow \mathcal{S}p$, the *nth cross effect* $\mathrm{cross}_n F(X_1, \dots, X_n)$ is defined as in [21] to be the total fiber of the cube

$$S \subset \underline{n} \quad \rightsquigarrow \quad F\left(\bigcup_{i \in S} X_i\right)$$

whose maps come from inclusions of subsets of $\underline{n}$. Here the big union denotes pushout along B; one can think of it as a fiberwise wedge sum. Since F is contravariant, the initial vertex of this cube corresponds to the full subset $S = \underline{n}$. Note that there is a natural map

$$\mathrm{cross}_n F(X_1, \dots, X_n) \xrightarrow{i_n} F\left(\bigcup_{i \in \underline{n}} X_i\right)$$

Similarly, the *nth co-cross effect* $\mathrm{cocross}_n F(X_1, \dots, X_n)$ is defined as in [30] and [6] to be the total cofiber of the cube with the same vertices

$$S \subset \underline{n} \quad \rightsquigarrow \quad F\left(\bigcup_{i \in S} X_i\right)$$

where the maps come from the opposites of inclusions of subsets of $\underline{n}$. Each inclusion $S \subseteq T$ results in a collapsing map

$$\bigcup_{i \in S} X_i \longleftarrow \bigcup_{i \in T} X_i$$

which becomes

$$F\left(\bigcup_{i \in S} X_i\right) \longrightarrow F\left(\bigcup_{i \in T} X_i\right)$$

Note that the final vertex of this cube corresponds to $S = \underline{n}$, so there is a natural map

$$F\left(\bigcup_{i\in\underline{n}} X_i\right) \xrightarrow{p_n} \mathrm{cocross}_n F(X_1,\dots,X_n)$$

It is known that the cross effect and co-cross effect are equivalent, when F is a functor from spectra to spectra ([6], Lemma 2.2). A similar argument gives the following.

Proposition 8.5. *If $\mathcal{R}_B^{\mathrm{op}} \xrightarrow{F} \mathcal{S}p$ is any contravariant functor, then the composite*

$$\mathrm{cross}_n F(X_1,\dots,X_n) \xrightarrow{i_n} F\left(\bigcup_{i\in\underline{n}} X_i\right) \xrightarrow{p_n} \mathrm{cocross}_n F(X_1,\dots,X_n)$$

is an equivalence. Furthermore, $F(\bigcup X_i)$ splits into a sum of cross-effects:

$$\begin{aligned} F\left(\bigcup_{i\in\underline{n}} X_i\right) &\simeq \prod_{S\subseteq\underline{n}} \mathrm{cocross}_{|S|} F(X_s : s\in S) \\ &\simeq \prod_{S\subseteq\underline{n}} \mathrm{cross}_{|S|} F(X_s : s\in S) \\ &\simeq \bigvee_{S\subseteq\underline{n}} \mathrm{cross}_{|S|} F(X_s : s\in S) \end{aligned}$$

The analogous result also holds for covariant functors, and for functors defined only on finite CW complexes.

Remark. This does *not* assume that F is a homotopy functor.

The argument is by induction on n. We form the maps

$$\bigvee_{S\subsetneq\underline{n}} \mathrm{cross}_{|S|} F(X_s : s\in S) \longrightarrow F\left(\bigcup_{i\in\underline{n}} X_i\right) \longrightarrow \prod_{S\subsetneq\underline{n}} \mathrm{cocross}_{|S|} F(X_s : s\in S)$$

and observe that the composite is an equivalence. Therefore the middle contains either of the outside terms as a summand. We use the alternate definitions of tfib and tcofib found in [20] to identify the leftover summand with $\mathrm{cross}_{|S|}F$ and $\mathrm{cocross}_{|S|}F$, which proves that they are equivalent and that F splits into a sum of cross effects.

This is a generalization of the following well known result: (cf. [4], [10])

Corollary 8.6 (Binomial Theorem for Suspension Spectra)**.** *If X and Y are well-based spaces then the obvious projection maps yield a splitting*

$$\Sigma^\infty(X\times Y) \xrightarrow{\sim} \Sigma^\infty(X\wedge Y)\times\Sigma^\infty X\times\Sigma^\infty Y$$

If X_1, ..., X_n are well-based spaces then we get a more general splitting

$$\Sigma^\infty \prod_{i=1}^n X_i \xrightarrow{\sim} \prod_{\emptyset\neq S\subset\underline{n}} \Sigma^\infty \bigwedge_{i\in S} X_i$$

and in particular if X is well-based then

$$\Sigma^\infty X^n \simeq \bigvee_{i=1}^{n} \binom{n}{i} \Sigma^\infty X^{\wedge i}$$

Remark. The corollary also follows easily if we start with

$$\Sigma^\infty(X_+) \simeq \Sigma^\infty(X \vee S^0)$$

From there the proof is suggested by the facts

$$\begin{aligned} (x+1)(y+1) - 1 &= xy + x + y \\ (x+1)^n - 1 &= \sum_{i=1}^{n} \binom{n}{i} x^i \end{aligned}$$

We are now in a position to prove the existence of $P_n F$ for retractive spaces into spectra. First we'll give a result that motivates the construction.

8.1. **An Equivalence Between $[\mathbf{G}_n^{\mathbf{op}}, \mathcal{S}p]$ and $[\mathbf{M}_n^{\mathbf{op}}, \mathcal{S}p]$.** Let $\mathbf{G}_n$ be the category of based sets $\underline{0}_+, \dots, \underline{n}_+$ and based maps between them; this is the opposite category of Segal's category Γ. Let $\mathbf{M}_n$ be the category of unbased sets $\underline{0} = \emptyset$, $\underline{1}$, $\dots$, $\underline{n}$ and *surjective* maps between them. If $\mathbf{I}$ is a category then let $[\mathbf{I}, \mathcal{S}p]$ denote the homotopy category of diagrams of spectra indexed by $\mathbf{I}$.

Note that the maps in $\mathbf{G}_n$ are generated by inclusions, collapses, rearrangements, and maps that fold two points into one. From the last section, a diagram of spectra indexed by $\mathbf{G}_n$ will split into a sum of cross effects. The first two classes of maps (inclusions and collapses) will simply include or collapse these summands. Therefore our diagram has redundancies. If we throw out the redundancies, only the last two classes of maps (rearrangements and folds) still carry interesting information. But these are exactly the maps that generate the smaller category $\mathbf{M}_n$. We have just given a heuristic argument for the following known result:

Proposition 8.7. *There is an equivalence of homotopy categories*

$$[\mathbf{G}_n, \mathcal{S}p] \xrightarrow{C} [\mathbf{M}_n, \mathcal{S}p]$$

obtained by taking cross-effects

$$CF(\underline{i}) = \mathrm{cross}_i F(\underline{1}_+, \dots, \underline{1}_+)$$

Its inverse is obtained by taking sums

$$[\mathbf{G}_n, \mathcal{S}p] \xleftarrow{P} [\mathbf{M}_n, \mathcal{S}p]$$

$$PG(\underline{i}_+) = \bigvee_{j=0}^{i} \binom{i}{j} G(j)$$

There is also an equivalence of homotopy categories

$$[\mathbf{G}_n^{\mathrm{op}}, \mathcal{S}p] \simeq [\mathbf{M}_n^{\mathrm{op}}, \mathcal{S}p]$$

obtained from co-cross effects and products

$$CF(\underline{i}) = \mathrm{cocross}_i F(\underline{1}_+, \ldots, \underline{1}_+)$$

$$PG(\underline{i}_+) = \prod_{j=0}^{i} \binom{i}{j} G(j)$$

Remark. The author learned a version of this result from Greg Arone. A similar result for diagrams of abelian groups was done by Pirashvili [32]. Helmstutler [24] gives a more sophisticated treatment that handles both abelian groups and spectra in the same uniform way. He gives a Quillen equivalence between the two categories of diagrams with the projective model structure. This is of course stronger than just an equivalence of homotopy categories. The above result gives an explicit description of the derived functors of Helmstutler's Quillen equivalence. This perspective was essential in making the correct guess for $P_n F$ in section 3 above, and it motivates our proof of Thm. 8.8 below.

Proof. We define diagrams that extend the above constructions on objects. The essential ingredient is to define maps between the various cubes that show up in the definition of total homotopy fiber and cofiber found in [20]. These maps of cubes $I^{\underline{i}} \longrightarrow I^{\underline{j}}$ are all generalized diagonal maps coming from maps of sets $\underline{i} \longleftarrow \underline{j}$. Then it is easy to define a natural equivalence of diagrams $CPG \longrightarrow G$. On the other hand, Prop. 8.5 gives an equivalence $PCF(\underline{i}_+) \longrightarrow F(\underline{i}_+)$ for each object $\underline{i}_+ \in \mathbf{G}_n$, but these equivalences do not commute with the maps of $\mathbf{G}_n$. Instead, we define an isomorphism $PCF \longrightarrow F$ in the homotopy category of diagrams. To do this, we choose for each arrow $\underline{i}_+ \longrightarrow \underline{j}_+$ of $\mathbf{G}_n$ a contractible space of maps

$$PCF(\underline{i}_+) \longrightarrow F(\underline{j}_+)$$

that agrees in a natural way with compositions, and such that on the identity arrows $\underline{i}_+ = \underline{i}_+$ we choose only equivalences

$$PCF(\underline{i}_+) \longrightarrow F(\underline{i}_+)$$

Our chosen spaces of maps $PCF(\underline{i}_+) \longrightarrow F(\underline{j}_+)$ end up being products of cubes, the same cubes that appear in the definition of total homotopy fiber above. This gives the desired equivalence of homotopy categories.

The contravariant case is similar, but we will give one more detail here since it is needed in the next section. If $F : \mathbf{G}_n^{\mathrm{op}} \longrightarrow \mathcal{S}p$, we use the diagonal maps $I^{\underline{i}} \longleftarrow I^{\underline{j}}$ to define

$$\bigvee_{S \subset \underline{i}} I_+^{\underline{i}-S} \wedge F(S_+) \longrightarrow \bigvee_{T \subset \underline{j}} I_+^{\underline{j}-T} \wedge F(T_+)$$

taking the summand for $S \subset \underline{i}$ to the summand for $f^{-1}(S) \subset \underline{j}$. This passes to a well-defined map on the co-cross effects of F, which gives the arrows of the diagram CF. $\square$

8.2. P_nF **for Retractive Spaces over** B **into Spectra.** Consider homotopy functors F from finite retractive spaces $\mathcal{R}_{B,\mathrm{fin}}^{\mathrm{op}}$ into spectra. Our previous construction of P_nF was roughly the same as a mapping space of diagrams indexed by $\mathbf{U}_{B,n}^{\mathrm{op}}$, the spaces under B with at most n points. When $B \neq *$, this approach calls for more technology because $\mathbf{U}_{B,n}$ needs to be enriched. However, the equivalence $[\mathbf{G}_n^{\mathrm{op}}, \mathcal{S}p] \simeq [\mathbf{M}_n^{\mathrm{op}}, \mathcal{S}p]$ suggests that we could just eliminate the inclusion and collapse maps in $\mathbf{U}_{B,n}$. This leads to the category $\mathbf{M}_n$ again, which does not need to be enriched.

So we replace our diagrams

$$\begin{gathered} \mathbf{U}_{B,n} \longrightarrow \mathcal{S}p \\ \underline{i}_B \rightsquigarrow X^i \\ \underline{i}_B \rightsquigarrow F(\underline{i}_B) \end{gathered}$$

with the diagrams of co-cross effects

$$\begin{gathered} \mathbf{M}_n \longrightarrow \mathcal{S}p \\ \underline{i} \rightsquigarrow X^{\overline{\wedge} i} \\ \underline{i} \rightsquigarrow \mathrm{cocross}_i F(\underline{1}_B, \ldots, \underline{1}_B) \end{gathered}$$

where $\overline{\wedge}$ is the external smash product from Def. 3.2. We are being sloppy about the existence of maps into B^i, but this gives enough intuition to suggest that we try the following construction on retractive simplicial sets $X_\cdot$ over $S_\cdot B$:

$$\begin{aligned} E_nF(X_\cdot) &= \operatorname*{holim}_{(\underline{i} \longleftarrow \underline{j}) \in a\mathbf{M}_n^{\mathrm{op}}} \left(\operatorname*{holim}_{\Delta_{X_\cdot^{\overline{\wedge} i}}} \mathrm{cocross}_j F(\Delta_B^p, \ldots, \Delta_B^p) \right) \\ &\simeq \operatorname*{holim}_{\mathbf{M}_n^{\mathrm{op}} \int \Delta_{X_\cdot^{\overline{\wedge} i}}} \mathrm{cocross}_i F(\Delta_B^p, \ldots, \Delta_B^p) \end{aligned}$$

As before, the equivalence comes from Prop. 6.8. Here $X_\cdot^{\overline{\wedge} i}$ is a simplicial set containing $(S_\cdot B)^i$ as a retract, whose fiber over a simplex in $(S_\cdot B)^i$ is the smash product of the fibers in $X_\cdot$. The homotopy type of $X_\cdot^{\overline{\wedge} i}$ is homotopy invariant in $X_\cdot$ by the same argument as Prop. 3.7 above. As before, we extend E_nF to spaces by $E_nF(X) := E_nF(S_\cdot X)$.

Each surjective map $\underline{i} \longleftarrow \underline{j}$ induces a cofibration $X_\cdot^{\overline{\wedge} i} \longrightarrow X_\cdot^{\overline{\wedge} j}$. This determines a functor $\Delta : \mathbf{M}_n^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ by that sends $\underline{i}$ to the category $\Delta_{X_\cdot^{\overline{\wedge} i}}$. The diagram

$$\mathbf{M}_n^{\mathrm{op}} \int \Delta_{X^{\overline{\wedge} i}} \overset{\mathrm{cocross}F}{\longrightarrow} \mathcal{S}p$$

is then defined by

$$\begin{tikzcd} \underline{i}, & \Delta[p] \arrow[r] & X_\cdot^{\overline{\wedge} i} \arrow[d] & \rightsquigarrow & \mathrm{cocross}_i F(\Delta_B^p, \ldots, \Delta_B^p) \arrow[d] \\ \underline{j}, \arrow[u] & \Delta[q] \arrow[r] \arrow[u] & X_\cdot^{\overline{\wedge} j} & \rightsquigarrow & \mathrm{cocross}_j F(\Delta_B^q, \ldots, \Delta_B^q) \end{tikzcd}$$

The map of co-cross effects is defined in the proof of Prop. 8.7 above. We can show that $E_n F$ is n-excisive by proving properties (1), (2), and (3) from section 7. Property (1) follows from the above equivalences, and property (3) is straightforward. We will do (2) in detail.

As before, we can start with a pushout cube of simplicial sets, with initial vertex $A \in \mathbf{sSet}_{S.B}$. It's indexed by a set S, so for each element $s \in S$, let $X_s \in \mathbf{sSet}_{S.B}$ be a simplicial set containing A (and also containing $S.B$ as a retract). Then there is a pushout cube which assigns each subset $T \subset S$ to the simplicial set $\bigcup_{t \in T} X_t$, which is shorthand for the pushout of the X_t along A. We want to show that $E_n F$ takes this to a Cartesian cube; in other words, the natural map

$$\operatorname*{holim}_{\Delta[p] \to (\bigcup_S X_s)^{\overline{\wedge}\, i}} \mathrm{cocross}_i F(\Delta^p_B, \ldots, \Delta^p_B) \longrightarrow \operatorname*{holim}_{(T \subsetneq S)^{\mathrm{op}}} \left(\operatorname*{holim}_{\Delta[p] \to (\bigcup_T X_s)^{\overline{\wedge}\, i}} \mathrm{cocross}_i F(\Delta^p_B, \ldots, \Delta^p_B) \right)$$

is an equivalence. Using dual Thomason, we rewrite the right-hand side as

$$\operatorname*{holim}_{(T, \underline{i}, \Delta[p] \to (\bigcup_T X_s)^{\overline{\wedge}\, i})} \mathrm{cocross}_i F(\Delta^p_B, \ldots, \Delta^p_B)$$

where each object of the indexing category is a proper subset $T \subsetneq S$, integers $p \geq 0$ and $0 \leq i \leq n$, and a map $\Delta[p] \to (\bigcup_T X_s)^{\overline{\wedge}\, i}$. A map between two objects looks like

$$\begin{tikzcd}
T, & \underline{i}, & \Delta[p] \arrow[r] & (\bigcup_T X_s)^{\overline{\wedge}\, i} \arrow[d] \\
 & & & (\bigcup_T X_s)^{\overline{\wedge}\, j} \\
U, \arrow[uu, "\text{subset}"] & \underline{j}, \arrow[uu] & \Delta[q] \arrow[r] \arrow[uu] & (\bigcup_U X_s)^{\overline{\wedge}\, j} \arrow[u]
\end{tikzcd}$$

As before, this category maps forward into $\mathbf{M}_n^{\mathrm{op}} \int \Delta_{(\bigcup_S X_s)^{\overline{\wedge}\, i}}$, in which a map between two objects looks like

$$\begin{tikzcd}
\underline{i}, & \Delta[p] \arrow[r] & (\bigcup_S X_s)^{\overline{\wedge}\, i} \arrow[d, leftrightarrow] \\
\underline{j}, \arrow[u] & \Delta[q] \arrow[r] \arrow[u] & (\bigcup_S X_s)^{\overline{\wedge}\, j}
\end{tikzcd}$$

This functor α forgets the data of T and includes $(\bigcup_T X_s)^{\overline{\wedge}\, i}$ into $(\bigcup_S X_s)^{\overline{\wedge}\, i}$. The natural map of homotopy limits defined above is again induced by a pullback of diagrams along α, so we just have to show that α is homotopy initial. Given an object $\Delta[q] \xrightarrow{\varphi} (\bigcup_S X_s)^{\overline{\wedge}\, j}$ in the target category, the overcategory $(\alpha \downarrow \varphi)$ has as its objects the factorizations of φ

$$\Delta[q] \longrightarrow \Delta[p] \longrightarrow (\bigcup_T X_s)^{\overline{\wedge}\, i} \longrightarrow (\bigcup_S X_s)^{\overline{\wedge}\, j}$$

where $T \subsetneq S$ must be a *proper* subset of S.

Let us give a terminal object for this overcategory. Either the map out of Δ^q hits the basepoint section, in which case we take $T = \emptyset$, or it misses the basepoint section, in which case it gives a j-tuple of simplices in $\bigcup_S X_s$, each of which lands *inside* one of the sets X_s in the pushout. Therefore there is a smallest subset $T \subset S$ such that $\Delta[q] \xrightarrow{\varphi} (\bigcup_S X_s)^{\overline{\wedge} j}$ lands inside $(\bigcup_T X_s)^{\overline{\wedge} j}$, and since $j \leq n < |S|$, this subset is proper. This gives a terminal object for the overcategory $(\alpha \downarrow \varphi)$, so it's contractible, which finishes (2).

We might now expect that $F \longrightarrow E_n F$ is an equivalence on $\mathbf{R}^{\mathrm{op}}_{B,n}$. This is actually not true. To fix it, define

$$P_n F(X) = \overline{E_n F}(X) \times F(0_B)$$

Corollary 8.4 above assures us that this isn't a terrible idea. Note that $P_n F(X)$ is n-excisive because it is a homotopy limit of n-excisive functors.

Now let $X = \underline{j}_B$. Then $X^{\overline{\wedge} i} \cong (\underline{j})^i_B$. We can partition $\Delta_{X^{\overline{\wedge} i}}$ into two categories, one in which the simplex lands in the basepoint section and another in which the simplex misses the basepoint section. This leads to a partition of $\mathbf{M}^{\mathrm{op}}_n \int \Delta$ into three categories, one in which there are no simplices, one in which the simplices land in B, and one in which the simplices miss B. The homotopy limit of the first two is $E_n F(0_B)$, which contains the homotopy limit of the first $F(0_B)$. So the homotopy limit of the last category is therefore $\overline{E_n F}(\underline{j}_B)$. This last category contains a homotopy initial subcategory of objects $\Delta[0] \times \underline{i} \hookrightarrow \underline{j}$, with $i \neq 0$ and $\underline{i} \hookrightarrow \underline{j}$ an order-preserving inclusion. Therefore

$$\overline{E_n F}(\underline{j}_B) \simeq \operatorname*{holim}_{\underline{0} \neq \underline{i} \hookrightarrow \underline{j}} \mathrm{cocross}_i F(1_B, \ldots, 1_B)$$

But the only surjective maps between subsets of $\underline{j}$ that respect the inclusion into $\underline{j}$ are identity maps. So this homotopy limit is an ordinary product:

$$\overline{E_n F}(\underline{j}_B) \simeq \prod_{\emptyset \neq \underline{i} \subset \underline{j}} \mathrm{cocross}_i F(1_B, \ldots, 1_B)$$

$$P_n F(\underline{j}_B) \simeq \prod_{\underline{i} \subset \underline{j}} \mathrm{cocross}_i F(1_B, \ldots, 1_B)$$

Using our splitting result (Prop. 8.5), this shows that $F(\underline{j}_B) \longrightarrow P_n F(\underline{j}_B)$ is an equivalence. This finishes the proof that $P_n F$ exists for F from retractive spaces over B into spectra:

Theorem 8.8. *If $F : \mathcal{R}^{\mathrm{op}}_{B,\mathrm{fin}} \longrightarrow \mathcal{S}p$ is a homotopy functor, then there is a universal n-excisive approximation $P_n F$, and $F \longrightarrow P_n F$ is an equivalence on spaces with at most n points.*

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